

ALGEBRAIC MONTGOMERY-YANG PROBLEM: THE NON-RATIONAL CASE AND THE DEL PEZZO CASE

DONGSEON HWANG AND JONGHAE KEUM

ABSTRACT. Montgomery-Yang problem predicts that every pseudofree circle action on the 5-dimensional sphere has at most 3 non-free orbits. Using a certain one-to-one correspondence, Kollár formulated the algebraic version of the Montgomery-Yang problem: every projective surface S with the second Betti number $b_2(S) = 1$ and with quotient singularities has at most 3 singular points if its smooth locus S^0 is simply-connected. In a previous paper, we have confirmed the conjecture when S has at least one non-cyclic quotient singularity. In this paper, we prove the conjecture either when S is not rational or when $-K_S$ is ample. Thus the conjecture is reduced to the case where S is a rational surface with K_S ample having at worst cyclic singularities.

1. INTRODUCTION

A pseudofree \mathbb{S}^1 -action on a sphere \mathbb{S}^{2k-1} is a smooth \mathbb{S}^1 -action which is free except for finitely many non-free orbits (whose isotropy types $\mathbb{Z}_{m_1}, \dots, \mathbb{Z}_{m_n}$ have pairwise relatively prime orders).

For $k = 2$ Seifert [Se] showed that such an action must be linear and hence has at most two non-free orbits. In the contrast to this, for $k = 4$ Montgomery and Yang [MY] showed that given any pairwise relatively prime collection of positive integers m_1, \dots, m_n , there is a pseudofree \mathbb{S}^1 -action on homotopy 7-sphere whose non-free orbits have exactly those orders. Petrie [P] proved similar results in all higher odd dimensions. This led Fintushel and Stern to formulate the following problem:

Conjecture 1.1 ([FS87]). (Montgomery-Yang Problem)

Let

$$\mathbb{S}^1 \times \mathbb{S}^5 \rightarrow \mathbb{S}^5$$

be a pseudo-free \mathbb{S}^1 -action. Then it has at most 3 non-free orbits.

The problem has remained unsolved since its formulation.

Pseudofree \mathbb{S}^1 -actions on 5-manifolds L have been studied in terms of the 4-dimensional quotient orbifold L/\mathbb{S}^1 (see e.g., [FS85], [FS87]). The following one-to-one correspondence was known to Montgomery, Yang, Fintushel and Stern, and recently observed by Kollár ([Kol05], [Kol08]):

Theorem 1.2 (cf. [Kol05], [Kol08]). *There is a one-to-one correspondence between:*

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- (1) *Pseudofree \mathbb{S}^1 -actions on 5 dimensional rational homology spheres L with $H_1(L, \mathbb{Z}) = 0$.*
- (2) *Smooth, compact 4 manifolds M with boundary such that*
 - (a) *$\partial M = \cup_i L_i$ is a disjoint union of lens spaces $L_i = \mathbb{S}^3/\mathbb{Z}_{m_i}$,*
 - (b) *the m_i are relatively prime to each other,*
 - (c) *$H_1(M, \mathbb{Z}) = 0$ and $H_2(M, \mathbb{Z}) \cong \mathbb{Z}$.*

Furthermore, L is diffeomorphic to \mathbb{S}^5 iff $\pi_1(M) = 1$.

We recall that a normal projective surface with the same Betti numbers with the projective plane \mathbb{CP}^2 is called a *rational homology projective plane*, a \mathbb{Q} -homology projective plane or a \mathbb{Q} -homology \mathbb{CP}^2 . When a normal projective surface S has quotient singularities only, S is a \mathbb{Q} -homology projective plane if the second Betti number $b_2(S) = 1$.

It is known that a \mathbb{Q} -homology projective plane with quotient singularities has at most 5 singular points (cf. [HK1] Corollary 3.4). Recently, the authors have classified \mathbb{Q} -homology projective planes with 5 quotient singularities ([HK1], also see [Keu10]).

Using the one-to-one correspondence, Kollár formulated the algebraic version of the Montgomery-Yang problem as follows:

Conjecture 1.3 ([Kol08]). (Algebraic Montgomery-Yang Problem)

Let S be a \mathbb{Q} -homology projective plane with quotient singularities. Assume that $S^0 := S \setminus \text{Sing}(S)$ is simply-connected. Then S has at most 3 singular points.

In a previous paper [HK2], we have confirmed the conjecture when S has at least one non-cyclic quotient singularity.

In this paper, we consider the case where S has cyclic singularities only. We first verify the conjecture when S is not rational.

Theorem 1.4. *Let S be a \mathbb{Q} -homology projective plane with cyclic singularities. Assume that $H_1(S^0, \mathbb{Z}) = 0$. If S is not rational, then S has at most 3 singular points.*

Remark 1.5. The condition $H_1(S^0, \mathbb{Z}) = 0$ is weaker than the condition $\pi_1(S^0) = \{1\}$, and there are examples of \mathbb{Q} -homology projective planes with 4 quotient singularities, not all cyclic, such that $H_1(S^0, \mathbb{Z}) = 0$. Such surfaces are completely classified in [HK2]. It turns out that they are log del Pezzo surfaces with 3 cyclic singularities and 1 non-cyclic singularity such that $H_1(S^0, \mathbb{Z}) = 0$ but $\pi_1(S^0) \cong \mathfrak{A}_5$, the simple group of order 60.

Next, we also prove the conjecture when $-K_S$ is ample.

Theorem 1.6. *Let S be a \mathbb{Q} -homology projective plane with cyclic singularities. Assume that $H_1(S^0, \mathbb{Z}) = 0$. If $-K_S$ is ample, then S has at most 3 singular points.*

Remark 1.7. (1) The condition $H_1(S^0, \mathbb{Z}) = 0$ implies that K_S is not numerically trivial, i.e., K_S or $-K_S$ is ample (Lemma 3.6). Thus, Theorems 1.4 and 1.6 together reduce Conjecture 1.3 to the case where S is a rational surface with cyclic singularities such that K_S is ample.

(2) Rational surfaces S with cyclic singularities have been studied extensively when $-K_S$ is ample or numerically trivial. In the former case the surface is called a log del Pezzo surface, and in the latter the surface is called a log Enriques surface. On the other hand, when K_S is ample, very little is known about the classification of

such surfaces. Moreover, if in addition $b_2(S) = 1$, that is, if S is a \mathbb{Q} -homology projective plane with K_S ample having at worst cyclic singularities, nothing seems to be known except the examples due to Kollár ([Kol08], Example 43). He constructed a series of such examples by contracting two rational curves on some well-chosen weighted projective hypersurfaces. Kollár's examples have $|Sing(S)| = 2$. In [HK3] we give new examples with $|Sing(S)| = 1, 2$, or 3 , all constructed geometrically, i.e., by blowing up the projective plane and then contracting chains of rational curves.

The proof of Theorem 1.4 goes as follows.

Let S be a \mathbb{Q} -homology projective plane with cyclic singularities such that $H_1(S^0, \mathbb{Z}) = 0$. Then the orders of local fundamental groups of singular points are pairwise relatively prime (Lemma 3.6). Also, by the orbifold Bogomolov-Miyaoka-Yau inequality (see Theorems 3.2, 3.3) S has at most 4 singular points. Assume that S has 4 singular points. Then the same inequality enables us to enumerate all possible 4-tuples consisting of the orders of local fundamental groups of singular points:

$$\begin{aligned} (2, 3, 5, q), & \quad q \geq 7, \quad \gcd(q, 30) = 1; \\ (2, 3, 7, q), & \quad 11 \leq q \leq 41, \quad \gcd(q, 42) = 1; \\ (2, 3, 11, 13). \end{aligned}$$

Given its minimal resolution

$$f : S' \rightarrow S,$$

the exceptional curves and the canonical class $K_{S'}$ span a sublattice

$$R + \langle K_{S'} \rangle$$

of the unimodular lattice

$$H^2(S', \mathbb{Z})_{free} := H^2(S', \mathbb{Z}) / (\text{torsion}),$$

where R is the sublattice spanned by the exceptional curves. We note that K_S is not numerically trivial (Lemma 3.6), hence $R + \langle K_{S'} \rangle$ is of finite index in $H^2(S', \mathbb{Z})_{free}$. As a consequence, its discriminant

$$D := |\det(R + \langle K_{S'} \rangle)|$$

is a positive square number (Lemma 3.6). This criterion significantly reduces the infinite list of all possible cases for R . For example, the order 3 singularity of the case $(2, 3, 5, q)$ must be of type $\frac{1}{3}(1, 1)$ (Lemma 5.3). The reduced list is still infinite, and almost all cases in the list cannot be ruled out by any further argument from lattice theory, e.g. computation of ϵ -invariants does not work here, which turned out to be effective in the proof of [HK1]. To handle this infinite list, we compute (-1) -curves on the minimal resolution S' . Assume further that S is not rational. This assumption implies that K_S is ample and S' contains a (-1) -curve E with $E \cdot (f^*K_S/K_S^2)$ small, i.e., with (f^*K_S/K_S^2) -degree small (Lemma 4.3). Then we proceed to prove that the existence of such a (-1) -curve E leads to a contradiction by using certain expressions of the intersection numbers $EK_{S'}$ and E^2 in terms of the intersection numbers of E with the exceptional curves and f^*K_S (Proposition 4.2). Here we also use the classification result for the case of 5 singular points [HK1].

The idea of computing (-1) -curves on the minimal resolution was first used in [Ke08] for some fixed types of singularities. In Proposition 4.2, we derive general formulas for arbitrary cyclic singularities. These formulas are useful in proving the non-existence of a curve on S' with prescribed intersection numbers with the exceptional curves.

The proof of Theorem 1.6 is given in Section 7 and 8. Here we also need, besides the previous ingredients, some detailed properties of del Pezzo surfaces of rank one with cyclic singularities developed by Zhang [Z], Gurjar and Zhang [GZ] and Belousov [Be].

Throughout this paper, we work over the field \mathbb{C} of complex numbers.

Notation

- $[n_1, n_2, \dots, n_l]$ a Hirzebruch-Jung continued fraction, i.e.,

$$[n_1, n_2, \dots, n_l] = n_1 - \frac{1}{n_2 - \frac{1}{\ddots - \frac{1}{n_l}}} = \frac{q}{q_1}$$

corresponding to a cyclic singularity of type $\frac{1}{q}(1, q_1)$.

- $||[n_1, n_2, \dots, n_l]| = q$.
- $b_i(X)$ the i -th Betti number of a complex variety X .
- $f : S' \rightarrow S$ a minimal resolution of a normal surface S .
- $Sing(S)$: the singular locus of S .
- $\mathcal{F} := f^{-1}(Sing(S))$ a reduced integral divisor on S' .
- R_p : the sublattice of $H^2(S', \mathbb{Z})_{free}$ spanned by the numerical classes of the components of $f^{-1}(p)$, where $H^2(S', \mathbb{Z})_{free} = H^2(S', \mathbb{Z})/(\text{torsion})$.
- $R := \bigoplus_{p \in Sing(S)} R_p$ the sublattice of $H^2(S', \mathbb{Z})_{free}$ spanned by the numerical classes of the irreducible exceptional curves of $f : S' \rightarrow S$.
- $L = L_S := \text{rank}(R)$, the number of the irreducible components of $\mathcal{F} = f^{-1}(Sing(S))$, or the number of the exceptional curves of $f : S' \rightarrow S$.

2. HIRZEBRUCH-JUNG CONTINUED FRACTIONS

Let \mathcal{H} be the set of all Hirzebruch-Jung continued fractions $[n_1, n_2, \dots, n_l]$,

$$\mathcal{H} = \bigcup_{l \geq 1} \{[n_1, n_2, \dots, n_l] \mid \text{all } n_j \text{ are integers } \geq 2\}.$$

Notation 2.1. Fix $w = [n_1, n_2, \dots, n_l] \in \mathcal{H}$.

- (1) The *length* of w , denoted by $l(w)$, is the number of entries of w .
- (2) The *trace* of w , $tr(w) = \sum_{j=1}^l n_j$, is the sum of entries of w .

(3) $|w| = |[n_1, n_2, \dots, n_l]| := |\det(M(-n_1, \dots, -n_l))|$, where

$$M(-n_1, \dots, -n_l) = \begin{pmatrix} -n_1 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & -n_2 & 1 & \cdots & \cdots & 0 \\ 0 & 1 & -n_3 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -n_{l-1} & 1 \\ 0 & 0 & 0 & \cdots & 1 & -n_l \end{pmatrix}$$

is the intersection matrix of $[n_1, n_2, \dots, n_l]$.

(4) $q := |w|$ = the order of the cyclic singularity corresponding to w , i.e., $w = \frac{q}{q_1}$ for some q_1 with $1 \leq q_1 < q$, $\gcd(q, q_1) = 1$.

$$q_{a_1, a_2, \dots, a_m} := |\det(M')|,$$

$$q_{1, 2, \dots, l} := |\det(M(\emptyset))| = 1,$$

where M' is the $(l-m) \times (l-m)$ matrix obtained by deleting $-n_{a_1}, -n_{a_2}, \dots, -n_{a_m}$ from $M(-n_1, \dots, -n_l)$. For example,

$$q_1 = |\det(M(-n_2, \dots, -n_l))| = |[n_2, n_3, \dots, n_l]|$$

$$q_l = |\det(M(-n_1, \dots, -n_{l-1}))| = |[n_1, n_2, \dots, n_{l-1}]|$$

$$q_{1, l} = |\det(M(-n_2, \dots, -n_{l-1}))| = |[n_2, n_3, \dots, n_{l-1}]|.$$

Note that

$$[n_l, n_{l-1}, \dots, n_1] = \frac{q}{q_l}$$

$$q_1 q_l = q_{1, l} q + 1 \text{ if } l \geq 2.$$

We will write simply l, tr for $l(w), tr(w)$ if there is no confusion.

The following number-theoretic property of Hirzebruch-Jung continued fractions will play a key role in the proof of Lemma 5.3.

Proposition 2.2. For $w = [n_1, n_2, \dots, n_l] \in \mathcal{H}$,

$$q_1 + q_l + tr \cdot q \not\equiv 0 \text{ modulo } 3 \text{ iff } q \equiv 0 \text{ modulo } 3.$$

Proof. In the following, $a \equiv b$ means that $a \equiv b$ modulo 3.

Assume $q \equiv 0$.

If $l = 1$ and $w = [n_1]$, then $q_1 = q_l = |\det(M(\emptyset))| = 1$ and $q = tr = n_1 \equiv 0$, hence

$$q_1 + q_l + tr \cdot q \equiv 1 + 1 + 0 \not\equiv 0.$$

If $l \geq 2$, then we see from the equality $q_1 q_l = q_{1, l} q + 1$ that $q_1 q_l \equiv 1$. Thus $q_1 \equiv q_l \equiv \pm 1$ and

$$q_1 + q_l + tr \cdot q \equiv \pm 1 \pm 1 + 0 \not\equiv 0.$$

Assume $q \not\equiv 0$, i.e., $q \equiv \pm 1$.

We will show by induction on l that

$$(2.1) \quad q_1 + q_l + tr \cdot q \equiv 0$$

If $l = 1$ and $w = [n_1]$, then $q_1 = q_l = 1$ and $q = tr = n_1 \equiv \pm 1$, hence

$$q_1 + q_l + tr \cdot q \equiv 1 + 1 + (\pm 1)^2 \equiv 0.$$

If $l = 2$ and $w = [n_1, n_2]$, then $q = n_1 n_2 - 1 \equiv \pm 1$, so $n_1 n_2 \equiv -1$ or 0 , hence $n_1 \equiv -n_2$ or $n_1 \equiv 0$ or $n_2 \equiv 0$. In any case,

$$q_1 + q_l + tr \cdot q = n_2 + n_1 + (n_1 + n_2)(n_1 n_2 - 1) = n_1 n_2 (n_1 + n_2) \equiv 0.$$

Now assume $l \geq 3$. We divide the proof into 3 cases $q_1 \equiv 1, -1, 0$.

Case (1): $q_1 \equiv 1$. By the induction hypothesis (2.1) holds for $[n_2, \dots, n_l]$, i.e.,

$$q_{1,2} + q_{1,l} + (tr - n_1) \cdot q_1 \equiv 0.$$

Plugging $q = n_1 q_1 - q_{1,2}$ into the above equality, we get

$$q_{1,l} + tr \cdot q_1 - q \equiv 0.$$

Thus

$$\begin{aligned} q_1 + q_l + tr \cdot q &\equiv 1 + q_l + tr \cdot q \\ &\equiv -1 - 1 + 1 \cdot q_l + tr \cdot q \\ &\equiv -1 - q^2 + q_1 q_l + tr \cdot q \\ &= q_{1,l} q + tr \cdot q - q^2 \\ &= (q_{1,l} + tr - q)q \\ &\equiv (q_{1,l} + tr \cdot q_1 - q)q \\ &\equiv 0. \end{aligned}$$

Case (2): $q_1 \equiv -1$. In this case, the induction hypothesis also gives $q_{1,l} + tr \cdot q_1 - q \equiv 0$. Thus

$$\begin{aligned} q_1 + q_l + tr \cdot q &\equiv -1 + q_l + tr \cdot q \\ &\equiv 1 - q_1 q_l + tr \cdot q + q^2 \\ &\equiv -q_{1,l} q - tr \cdot q_1 q + q^2 \\ &= -(q_{1,l} + tr \cdot q_1 - q)q \\ &\equiv 0. \end{aligned}$$

Case (3): $q_1 \equiv 0$. First note that $q = n_1 q_1 - q_{1,2} \equiv -q_{1,2}$, so $q_{1,2} \equiv -q \neq 0$. Also, note that $q_{1,l} q = q_1 q_l - 1 \equiv -1$, so $q_{1,l} \equiv -q$.

Since $q_{1,2} \neq 0$, we apply the induction hypothesis to $[n_3, \dots, n_l]$ to get

$$q_{1,2,3} + q_{1,2,l} + (tr - n_1 - n_2) \cdot q_{1,2} \equiv 0.$$

Note that $q_1 = n_2 q_{1,2} - q_{1,2,3}$ and $n_1 q_{1,l} - q_l = q_{1,2,l}$.

Since $q_{1,2} \equiv q_{1,l} \equiv -q$, we have

$$\begin{aligned} q_1 + q_l + tr \cdot q &\equiv q_1 + q_l - tr \cdot q_{1,2} \\ &\equiv q_1 - (n_1 q_{1,l} - q_l) - tr \cdot q_{1,2} + n_1 q_{1,2} \\ &= (n_2 q_{1,2} - q_{1,2,3}) - q_{1,2,l} - tr \cdot q_{1,2} + n_1 q_{1,2} \\ &= -q_{1,2,3} - q_{1,2,l} - (tr - n_1 - n_2) \cdot q_{1,2} \\ &\equiv 0. \end{aligned}$$

□

We collect some properties of Hirzebruch-Jung continued fractions which will be frequently used in the subsequent sections.

Notation 2.3. For a fixed continued fraction $w = [n_1, n_2, \dots, n_l] \in \mathcal{H}$ and an integer $0 \leq s \leq l+1$, we define

- (1) $u_s := q_{s, \dots, l} = |[n_1, n_2, \dots, n_{s-1}]|$ ($2 \leq s \leq l+1$), $u_0 = 0$, $u_1 = 1$
- (2) $v_s := q_{1, \dots, s} = |[n_{s+1}, n_{s+2}, \dots, n_l]|$ ($0 \leq s \leq l-1$), $v_l = 1$, $v_{l+1} = 0$.

Note that $u_l = q_l$, $u_{l+1} = q$, $v_0 = q$, $v_1 = q_1$.

Lemma 2.4. *Let $w = [n_1, n_2, \dots, n_l] \in \mathcal{H}$. Then,*

- (1) $u_{j+1} = n_j u_j - u_{j-1}$,
 $v_{j-1} = n_j v_j - v_{j+1}$.
- (2) $v_j u_{j+1} - v_{j+1} u_j = v_{j-1} u_j - v_j u_{j-1} = q$.
- (3) $v_j u_j = \frac{1}{n_j}(q + v_{j+1} u_j + v_j u_{j-1})$.
- (4) $\sum_{j=1}^s (n_j - 2) u_j = u_{s+1} - u_s - 1$,
 $\sum_{j=s}^l (n_j - 2) v_j = v_{s-1} - v_s - 1$.
- (5) $\frac{u_j + v_j}{q} \leq 1$.
- (6) $|[n_1, \dots, n_{j-1}, n_j + 1, n_{j+1}, \dots, n_l]| = u_j v_j + |[n_1, n_2, \dots, n_l]| > q$.

Proof. (1) is well-known.

(2) is obtained by a direct calculation using (1) as follows:

$$\begin{aligned}
 v_j u_{j+1} - v_{j+1} u_j &= (n_j u_j - u_{j-1}) v_j - v_{j+1} u_j \\
 &= (n_j v_j - v_{j+1}) u_j - v_j u_{j-1} \\
 &= v_{j-1} u_j - v_j u_{j-1} \\
 &\quad \dots \\
 &= v_1 u_2 - v_2 u_1 = q_1 n_1 - q_{1,2} = q.
 \end{aligned}$$

(3) follows from the equality

$$n_j v_j u_j = (v_{j-1} + v_{j+1}) u_j = q + v_j u_{j-1} + v_{j+1} u_j.$$

(4) follows from

$$\begin{aligned}
 (n_j - 2) u_j &= (u_{j+1} - u_j) - (u_j - u_{j-1}) \\
 (n_j - 2) v_j &= (v_{j+1} - v_j) - (v_j - v_{j-1}).
 \end{aligned}$$

(5) Note that

$$\begin{aligned}
 v_j &= n_{j+1} v_{j+1} - v_{j+2} \geq v_{j+1} + (v_{j+1} - v_{j+2}) \geq v_{j+1} + 1, \\
 u_{j+1} &= n_j u_j - u_{j-1} \geq u_j + (u_j - u_{j-1}) \geq u_j + 1.
 \end{aligned}$$

Thus

$$q - (v_j + u_j) = v_j(u_{j+1} - 1) - (v_{j+1} + 1)u_j \geq 0.$$

(6) Note that

$$|[n_1, \dots, n_{j-1}, n_j + 1]| = (n_j + 1)u_j - u_{j-1} = u_j + u_{j+1}.$$

By (2)

$$\begin{aligned}
 |[n_1, \dots, n_{j-1}, n_j + 1, n_{j+1}, \dots, n_l]| &= |[n_1, \dots, n_{j-1}, n_j + 1]| v_j - u_j v_{j+1} \\
 &= u_j v_j + u_{j+1} v_j - u_j v_{j+1} \\
 &= u_j v_j + |[n_1, n_2, \dots, n_l]|.
 \end{aligned}$$

□

Lemma 2.5. *Assume $l \geq 5$. Then for arbitrary non-negative integers z_1, \dots, z_l ,*

- (1) $\sum_{j=1}^l (u_j + v_j) z_j \leq \sum_{j=1}^l (u_j v_j) z_j^2$ when $\sum_{j=1}^l z_j \geq 3$,
- (2) $\sum_{j=1}^l (u_j + v_j) z_j \leq \sum_{j=1}^l (u_j v_j) z_j^2 + 2$ when $\sum_{j=1}^l z_j = 2$,

$$(3) \sum_{j=1}^l (u_j + v_j)z_j \leq \sum_{j=1}^l (u_j v_j)z_j^2 + 1 \text{ when } \sum_{j=1}^l z_j = 1.$$

Proof. Note that $(u_1 + v_1)z_1 = (1 + v_1)z_1 \leq v_1 z_1^2 - 2$ if $z_1 \geq 2$, and $(u_1 + v_1)z_1 = (1 + v_1)z_1 = v_1 z_1^2 + 1$ if $z_1 = 1$.

Similarly, $(u_l + v_l)z_l = (u_l + 1)z_l \leq u_l z_l^2 - 2$ if $z_l \geq 2$, and

$(u_l + v_l)z_l = (u_l + 1)z_l = u_l z_l^2 + 1$ if $z_l = 1$.

For $2 \leq j \leq l-1$, we have $u_j \geq 2, v_j \geq 2, u_j + v_j \geq 6$ since $l \geq 5$, so

$(u_j + v_j)z_j \leq (u_j v_j)z_j^2$ and $(u_j + v_j)z_j \leq (u_j v_j)z_j^2 - 2$ if $z_j \geq 1$. \square

3. ALGEBRAIC SURFACES WITH QUOTIENT SINGULARITIES

3.1. A singularity p of a normal surface S is called a quotient singularity if the germ is locally analytically isomorphic to $(\mathbb{C}^2/G, O)$ for some nontrivial finite subgroup G of $GL_2(\mathbb{C})$ without quasi-reflections. Brieskorn classified all such finite subgroups of $GL(2, \mathbb{C})$ [Bri].

Let S be a normal projective surface with quotient singularities and

$$f : S' \rightarrow S$$

be a minimal resolution of S . It is well-known that quotient singularities are log-terminal singularities. Thus one can write

$$K_{S'} \equiv_{num} f^* K_S - \sum_{p \in \text{Sing}(S)} \mathcal{D}_p,$$

where $\mathcal{D}_p = \sum (a_j A_j)$ is an effective \mathbb{Q} -divisor with $0 \leq a_j < 1$ supported on $f^{-1}(p) = \cup A_j$ for each singular point p . Intersecting the formula with \mathcal{D}_p , we get

$$\mathcal{D}_p K_{S'} = -\mathcal{D}_p^2$$

and hence

$$K_S^2 = K_{S'}^2 - \sum_p \mathcal{D}_p^2 = K_{S'}^2 + \sum_p \mathcal{D}_p K_{S'}.$$

For each singular point p , the coefficients of the \mathbb{Q} -divisor \mathcal{D}_p can be obtained by solving the equations given by the adjunction formula

$$\mathcal{D}_p A_j = -K_{S'} A_j = 2 + A_j^2$$

for each exceptional curve $A_j \subset f^{-1}(p)$.

When p is a cyclic singularity of order q , the coefficients of \mathcal{D}_p can be expressed in terms of v_j and u_j (see Notation 2.3) as follows.

Lemma 3.1. *Let p be a cyclic quotient singular point of S . Assume that $f^{-1}(p)$ has l components A_1, \dots, A_l with $A_i^2 = -n_i$ forming a string of smooth rational curves $\overset{-n_1}{\circ} - \overset{-n_2}{\circ} - \dots - \overset{-n_l}{\circ}$. Then*

$$(1) \mathcal{D}_p = \sum_{j=1}^l \left(1 - \frac{v_j + u_j}{q}\right) A_j,$$

$$(2) \mathcal{D}_p K_{S'} = -\mathcal{D}_p^2 = \sum_{j=1}^l \left(1 - \frac{v_j + u_j}{q}\right) (n_j - 2),$$

$$(3) \mathcal{D}_p^2 = 2l - \sum_{j=1}^l n_j + 2 - \frac{q_1 + q_l + 2}{q}.$$

In particular, if $l = 1$, then $\mathcal{D}_p^2 = -\frac{(n_1 - 2)^2}{n_1}$.

Proof. (1) is well known (cf. [Me] or Lemma 2.2 of [HK1]).

(2) follows from (1) and the adjunction formula.

(3) is also well known (cf. [LW] or Lemma 3.6 of [HK1]). \square

Also we recall the orbifold Euler characteristic

$$e_{orb}(S) := e(S) - \sum_{p \in \text{Sing}(S)} \left(1 - \frac{1}{|G_p|}\right),$$

where G_p is the local fundamental group of p .

The following theorem, called the orbifold Bogomolov-Miyaoka-Yau inequality, is one of the main ingredients in the proof of our main theorems.

Theorem 3.2 ([S], [Mi], [KNS], [Me]). *Let S be a normal projective surface with quotient singularities such that K_S is nef. Then*

$$K_S^2 \leq 3e_{orb}(S).$$

In particular,

$$0 \leq e_{orb}(S).$$

The weaker inequality holds when $-K_S$ is nef.

Theorem 3.3 ([KM]). *Let S be a normal projective surface with quotient singularities such that $-K_S$ is nef. Then*

$$0 \leq e_{orb}(S).$$

3.2. Let S be a normal projective surface with quotient singularities and $f : S' \rightarrow S$ be a minimal resolution of S . It is well-known that the torsion-free part of the second cohomology group,

$$H^2(S', \mathbb{Z})_{free} := H^2(S', \mathbb{Z}) / (\text{torsion})$$

has a lattice structure which is unimodular. For a quotient singular point $p \in S$, let

$$R_p \subset H^2(S', \mathbb{Z})_{free}$$

be the sublattice of $H^2(S', \mathbb{Z})_{free}$ spanned by the numerical classes of the components of $f^{-1}(p)$. It is a negative definite lattice, and its discriminant group

$$\text{disc}(R_p) := \text{Hom}(R_p, \mathbb{Z}) / R_p$$

is isomorphic to the abelianization $G_p/[G_p, G_p]$ of the local fundamental group G_p . In particular, the absolute value $|\det(R_p)|$ of the determinant of the intersection matrix of R_p is equal to the order $|G_p/[G_p, G_p]|$. Let

$$R = \bigoplus_{p \in \text{Sing}(S)} R_p \subset H^2(S', \mathbb{Z})_{free}$$

be the sublattice of $H^2(S', \mathbb{Z})_{free}$ spanned by the numerical classes of the exceptional curves of $f : S' \rightarrow S$. We also consider the sublattice

$$R + \langle K_{S'} \rangle \subset H^2(S', \mathbb{Z})_{free}$$

spanned by R and the canonical class $K_{S'}$. Note that

$$\text{rank}(R) \leq \text{rank}(R + \langle K_{S'} \rangle) \leq \text{rank}(R) + 1.$$

Lemma 3.4 ([HK1], Lemma 3.3). *Let S be a normal projective surface with quotient singularities and $f : S' \rightarrow S$ be a minimal resolution of S . Then the following hold true.*

- (1) $\text{rank}(R + \langle K_{S'} \rangle) = \text{rank}(R)$ if and only if K_S is numerically trivial.
- (2) $\det(R + \langle K_{S'} \rangle) = \det(R) \cdot K_S^2$ if K_S is not numerically trivial.
- (3) If in addition $b_2(S) = 1$ and K_S is not numerically trivial, then $R + \langle K_{S'} \rangle$ is a sublattice of finite index in the unimodular lattice $H^2(S', \mathbb{Z})_{\text{free}}$, in particular $|\det(R + \langle K_{S'} \rangle)|$ is a nonzero square number.

We denote the number $|\det(R + \langle K_{S'} \rangle)|$ by D , i.e., we define

$$D := |\det(R + \langle K_{S'} \rangle)|.$$

The following is well known.

Lemma 3.5. *Assume that p is a cyclic singularity such that $f^{-1}(p)$ has l components A_1, \dots, A_l with $A_i^2 = -n_i$ forming a string of smooth rational curves $\overset{-n_1}{\circ} - \overset{-n_2}{\circ} - \dots - \overset{-n_l}{\circ}$. Then $\text{disc}(R_p)$ is a cyclic group generated by*

$$e_p := A_l^* = -\frac{1}{q} \sum_{i=1}^l u_i A_i$$

where $u_i = |\det[n_1, n_2, \dots, n_{i-1}]|$ as in Notation 2.3. It has the property that

$$e_p A_l = 1, \quad e_p A_j = 0 \quad (1 \leq j \leq l-1) \quad \text{and} \quad e_p^2 = -\frac{u_l}{q} = -\frac{q_l}{q}.$$

The following will be also useful in our proof.

Lemma 3.6 ([HK2], Lemma 2.5). *Let S be a \mathbb{Q} -homology projective plane with cyclic singularities such that $H_1(S^0, \mathbb{Z}) = 0$. Let $f : S' \rightarrow S$ be a minimal resolution. Then*

- (1) $H^2(S', \mathbb{Z})$ is torsion free, i.e., $H^2(S', \mathbb{Z}) = H^2(S', \mathbb{Z})_{\text{free}}$,
- (2) R is a primitive sublattice of the unimodular lattice $H^2(S', \mathbb{Z})$,
- (3) $\text{disc}(R)$ is a cyclic group, in particular, the orders $|G_p| = |\det(R_p)|$ are pairwise relatively prime,
- (4) K_S is not numerically trivial, i.e., K_S is either ample or anti-ample,
- (5) $D = |\det(R)| K_S^2$ and is a nonzero square number,
- (6) the Picard group $\text{Pic}(S')$ is generated over \mathbb{Z} by the exceptional curves and a \mathbb{Q} -divisor M of the form

$$M = \frac{1}{\sqrt{D}} f^* K_S + \sum_{p \in \text{Sing}(S)} b_p e_p$$

for some integers b_p , where e_p is the generator of $\text{disc}(R_p)$ as in Lemma 3.5.

Finally we generalize Lemma 3.6 to the case without the condition $H_1(S^0, \mathbb{Z}) = 0$. We will encounter this general situation later in our proof (see Sections 5 and 6).

Let S be a \mathbb{Q} -homology projective plane with cyclic singularities and $f : S' \rightarrow S$ be a minimal resolution. Denote by $\text{Pic}(S')_{\text{free}}$ the group of numerical equivalence classes of divisors, i.e.,

$$\text{Pic}(S')_{\text{free}} := \text{Pic}(S') / (\text{torsion}).$$

With the intersection pairing, $\text{Pic}(S')_{\text{free}}$ becomes a unimodular lattice isometric to $H^2(S', \mathbb{Z})_{\text{free}}$. Denote by

$$\bar{R} \subset \text{Pic}(S')_{\text{free}}$$

the primitive closure of $R \subset \text{Pic}(S')_{free}$, the sublattice spanned by the numerical equivalence classes of exceptional curves of f .

Lemma 3.7. *Let S be a \mathbb{Q} -homology projective plane with cyclic singularities and $f : S' \rightarrow S$ be a minimal resolution. Assume that K_S is not numerically trivial. Then the following hold true.*

- (1) $D = |\det(R)|K_S^2$ and is a nonzero square number.
- (2) $\text{disc}(\bar{R})$ is a cyclic group of order $|\det(\bar{R})| = \frac{|\det(R)|}{c^2}$ where c is the order of \bar{R}/R .
- (3) Define

$$D' := |\det(\bar{R})|K_S^2 = \frac{D}{c^2}.$$

Then $\text{Pic}(S')_{free}$ is generated over \mathbb{Z} by the numerical equivalence classes of exceptional curves, an element $T \in \text{Pic}(S')_{free}$ giving a generator of \bar{R}/R and a \mathbb{Q} -divisor of the form

$$M = \frac{1}{\sqrt{D'}} f^* K_S + z,$$

where z is a generator of $\text{disc}(\bar{R})$, hence of the form $z = \sum_{p \in \text{Sing}(S)} b_p e_p$ for

some integers b_p , where e_p is the generator of $\text{disc}(R_p)$ as in Lemma 3.5.

- (4) For each singular point p , denote by $A_{1,p}, A_{2,p}, \dots, A_{l_p,p}$ the exceptional curves of f at p and by q_p the order of the local fundamental group at p . Then every element $E \in \text{Pic}(S')_{free}$ can be written uniquely as

$$(3.1) \quad E = mM + \sum_{p \in \text{Sing}(S)} \sum_{i=1}^{l_p} a_{i,p} A_{i,p}$$

for some integer m and some $a_{i,p} \in \frac{1}{c}\mathbb{Z}$ for all i, p .

- (5) E is supported on $f^{-1}(\text{Sing}(S))$ if and only if $m = 0$. Moreover, if E is effective (modulo a torsion) and not supported on $f^{-1}(\text{Sing}(S))$, then $m > 0$ when K_S is ample, and $m < 0$ when $-K_S$ is ample.

Proof. (1) follows from Lemma 3.4.

(2) is well known.

(3) We slightly modify the proof of [HK2], Lemma 2.5. Here, R^\perp is generated by

$$v := \frac{\sqrt{D'}}{K_S^2} f^* K_S = \frac{|\det(\bar{R})|}{\sqrt{D'}} f^* K_S,$$

$\text{disc}(R^\perp)$ is generated by

$$\frac{1}{\sqrt{D'}} f^* K_S,$$

and

$$\text{Pic}(S')_{free}/(R^\perp \oplus \bar{R}) \subset \text{disc}(R^\perp \oplus \bar{R})$$

is an isotropic subgroup of order $|\det(\bar{R})|$ of $\text{disc}(R^\perp \oplus \bar{R})$, hence is generated by an element

$$M \in \text{disc}(R^\perp \oplus \bar{R})$$

of order $|\det(\bar{R})|$. Moreover M is the sum of a generator of $\text{disc}(R^\perp)$ and a generator of $\text{disc}(\bar{R})$, since $\text{Pic}(S')_{free}$ is unimodular. By replacing M by kM for a

suitable choice of an integer k , we get M of the desired form. We have shown that $\text{Pic}(S')_{\text{free}}$ is generated over \mathbb{Z} by v , \bar{R} and M . Note that

$$|\det(\bar{R})|M \equiv v \text{ modulo } \bar{R},$$

i.e., v is generated by M and \bar{R} . Finally \bar{R} is generated over \mathbb{Z} by R and T .

(4) By (3) E is a \mathbb{Z} -linear combination of M , T , and $A_{i,p}$. Since $cT \in R$, the result follows.

(5) The first assertion is obvious. For the second, note that

$$E(f^*K_S) = mM(f^*K_S) = \frac{m}{\sqrt{D'}}K_S^2.$$

□

4. CURVES ON THE MINIMAL RESOLUTION

Throughout this section, we denote by S a \mathbb{Q} -homology projective plane with cyclic singularities and by $f : S' \rightarrow S$ its minimal resolution, and assume that K_S is not numerically trivial. But we do not assume that $H_1(S^0, \mathbb{Z}) = 0$. So, the orders of singularities may not be pairwise relatively prime.

Let E be a divisor on S' . Then by Lemma 3.7(4), the numerical equivalence class of E can be written as the form (3.1). The coefficients of E in (3.1) and the intersection numbers $EA_{j,p}$ are related as follows. Here u_j and v_j are as in Notation 2.3.

Lemma 4.1. *Fix $p \in \text{Sing}(S)$. Then for $j = 1, \dots, l_p$*

$$\frac{u_{j,p}}{q_p}mb_p - a_{j,p} = \sum_{k=1}^j \frac{v_{j,p}u_{k,p}}{q_p}(EA_{k,p}) + \sum_{k=j+1}^{l_p} \frac{v_{k,p}u_{j,p}}{q_p}(EA_{k,p}).$$

Proof. Note that, by Lemma 3.5, for each $p \in \text{Sing}(S)$

$$MA_{j,p} = 0 \text{ for } j = 1, \dots, l_p - 1, \text{ and } MA_{l_p,p} = b_p.$$

We fix p and, for simplicity, omit the subscript p . Thus we obtain the following system of equalities:

$$\begin{aligned} EA_1 &= -n_1a_1 + a_2 \\ EA_2 &= a_1 - n_2a_2 + a_3 \\ EA_3 &= a_2 - n_3a_3 + a_4 \\ &\dots \\ EA_{l-1} &= a_{l-2} - n_{l-1}a_{l-1} + a_l \\ EA_l &= a_{l-1} - n_la_l + mb. \end{aligned}$$

It implies that

$$\begin{aligned} a_1 &= \frac{1}{n_1}a_2 - \frac{1}{n_1}EA_1 = \frac{u_1}{u_2}a_2 - \frac{1}{u_2}EA_1 \\ a_2 &= \frac{u_2}{u_3}a_3 - \frac{1}{u_3}EA_1 - \frac{u_2}{u_3}EA_2 \\ &\dots \\ a_j &= \frac{u_j}{u_{j+1}}a_{j+1} - \frac{1}{u_{j+1}}EA_1 - \dots - \frac{u_j}{u_{j+1}}EA_j \\ &\dots \\ a_{l-1} &= \frac{u_{l-1}}{u_l}a_l - \frac{1}{u_l}EA_1 - \dots - \frac{u_{l-1}}{u_l}EA_{l-1} \\ a_l &= \frac{u_l}{q}mb - \frac{1}{q}EA_1 - \dots - \frac{u_l}{q}EA_l = \frac{u_l}{q}mb - \sum_{k=1}^l \frac{v_l u_k}{q}EA_k. \end{aligned}$$

Plugging the last equation into the above equation for a_{l-1} , we obtain

$$\begin{aligned} a_{l-1} &= \frac{u_{l-1}}{u_l} \left(\frac{u_l}{q} mb - \frac{1}{q} EA_1 - \dots - \frac{u_l}{q} EA_l \right) - \frac{1}{u_l} EA_1 - \dots - \frac{u_{l-1}}{u_l} EA_{l-1} \\ &= \frac{u_{l-1}}{q} mb - \sum_{k=1}^{l-1} \frac{(u_{l-1} + q)u_k}{qu_l} EA_k - \frac{u_{l-1}}{q} EA_l. \end{aligned}$$

By Lemma 2.4(2),

$$u_{l-1} + q = v_l u_{l-1} + q = v_{l-1} u_l,$$

so the required equation for a_{l-1} follows.

Next, plugging the required equation for a_{l-1} into the above equation for a_{l-2} , we obtain the required equation for a_{l-2} . Others can be obtained similarly. \square

Now we express the intersection numbers $EK_{S'}$ and E^2 in terms of the intersection numbers $EA_{j,p}$ of E and the exceptional curves $A_{j,p}$.

Proposition 4.2. *Let E be a divisor on S' . Write (the numerical equivalence class of) E as the form (3.1). Then the following hold true.*

$$(1) \quad EK_{S'} = \frac{m}{\sqrt{D'}} K_S^2 - \sum_p \sum_{j=1}^{l_p} \left(1 - \frac{v_{j,p} + u_{j,p}}{q_p} \right) EA_{j,p}.$$

If $EA_{j,p} \geq 0$ for all p and j , then

$$EK_{S'} \leq \frac{m}{\sqrt{D'}} K_S^2 - \sum_p \sum_{j=1}^{l_p} \left(1 - \frac{2}{n_{j,p}} \right) EA_{j,p}.$$

$$(2) \quad E^2 = \frac{m^2}{D'} K_S^2 - \sum_p \sum_{j=1}^{l_p} \left(\sum_{k=1}^j \frac{v_{j,p} u_{k,p}}{q_p} (EA_{k,p}) + \sum_{k=j+1}^{l_p} \frac{v_{k,p} u_{j,p}}{q_p} (EA_{k,p}) \right) EA_{j,p}.$$

If $EA_{j,p} \geq 0$ for all p and j , then

$$E^2 \leq \frac{m^2}{D'} K_S^2 - \sum_p \sum_{j=1}^{l_p} \frac{v_{j,p} u_{j,p}}{q_p} (EA_{j,p})^2.$$

(3) If, for each $p \in \text{Sing}(S)$, E has a non-zero intersection number with at most 2 components of $f^{-1}(p)$, i.e., $EA_{j,p} = 0$ for $j \neq s_p, t_p$ for some s_p and t_p with $1 \leq s_p < t_p \leq l_p$, then

$$E^2 = \frac{m^2}{D'} K_S^2 - \sum_p \left(\frac{v_{s_p} u_{s_p}}{q_p} (EA_{s_p})^2 + \frac{v_{t_p} u_{t_p}}{q_p} (EA_{t_p})^2 + \frac{2v_{t_p} u_{s_p}}{q_p} (EA_{s_p})(EA_{t_p}) \right).$$

Proof. (1) Note that

$$K_{S'} = f^*(K_S) - \sum_{p \in \text{Sing}(S)} \sum_{j=1}^{l_p} \left(1 - \frac{v_{j,p} + u_{j,p}}{q_p} \right) A_{j,p}.$$

Intersecting both sides with E , we get

$$EK_{S'} = Ef^*(K_S) - \sum_p \sum_{j=1}^{l_p} \left(1 - \frac{v_{j,p} + u_{j,p}}{q_p} \right) EA_{j,p}.$$

Intersecting both sides of

$$E = mM + \sum_p \sum_{i=1}^{l_p} a_{i,p} A_{i,p}$$

with $f^*(K_S)$, we get

$$Ef^*(K_S) = mMf^*(K_S) = \frac{m}{\sqrt{D'}}f^*(K_S)^2 = \frac{m}{\sqrt{D'}}K_S^2.$$

This proves the equality.

Note that

$$\begin{aligned} n_j(v_j + u_j) &= (v_{j+1} + v_{j-1}) + (u_{j+1} + u_{j-1}) && \text{(Lemma 2.4(1))} \\ &= (u_{j+1} + v_{j+1}) + (u_{j-1} + v_{j-1}) \leq 2q && \text{(Lemma 2.4(5)).} \end{aligned}$$

Thus

$$\frac{v_{j,p} + u_{j,p}}{q_p} \leq \frac{2}{n_{j,p}}$$

for all p and j . This gives the inequality.

(2) Intersecting both sides of

$$E = mM + \sum_p \sum_{j=1}^{l_p} a_{j,p} A_{j,p}$$

with E , we get

$$E^2 = mEM + \sum_p \sum_{j=1}^{l_p} a_{j,p} EA_{j,p}.$$

Intersecting both sides of

$$M = \frac{1}{\sqrt{D'}}f^*K_S + \sum_p b_p e_p$$

with E , we get

$$\begin{aligned} mEM &= \frac{m}{\sqrt{D'}}Ef^*(K_S) + m\sum_p b_p Ee_p \\ &= \frac{m}{\sqrt{D'}}\frac{m}{\sqrt{D'}}K_S^2 + m\sum_p b_p (mMe_p + a_{l,p}) \\ &= \frac{m^2}{D'}K_S^2 + m\sum_p b_p (mb_p e_p^2 + a_{l,p}) \\ &= \frac{m^2}{D'}K_S^2 + m\sum_p b_p \left(-\frac{mb_p u_{l,p}}{q} + a_{l,p}\right) && \text{(Lemma 3.5)} \\ &= \frac{m^2}{D'}K_S^2 - m\sum_p b_p \left(\sum_{k=1}^{l_p} \frac{v_{l,p} u_{k,p}}{q} EA_{k,p}\right) && \text{(Lemma 4.1).} \end{aligned}$$

Thus

$$\begin{aligned} E^2 &= \frac{m^2}{D'}K_S^2 - m\sum_p b_p \left(\sum_{j=1}^{l_p} \frac{v_{l,p} u_{j,p}}{q} EA_{j,p}\right) + \sum_p \sum_{j=1}^{l_p} a_{j,p} EA_{j,p} \\ &= \frac{m^2}{D'}K_S^2 - \sum_p \sum_{j=1}^{l_p} \left(\frac{mb_p u_{j,p}}{q} - a_{j,p}\right) EA_{j,p}. \end{aligned}$$

Now the equality follows from Lemma 4.1.

If $EA_{j,p} \geq 0$ for all p and j , then

$$\sum_{k=1}^j \frac{v_{j,p} u_{k,p}}{q_p} (EA_{k,p}) + \sum_{k=j+1}^{l_p} \frac{v_{k,p} u_{j,p}}{q_p} (EA_{k,p}) \geq \frac{v_{j,p} u_{j,p}}{q_p} (EA_{j,p}),$$

so the inequality follows.

(3) If $EA_{j,p} = 0$ for $j \neq s_p, t_p$ for some s_p and t_p with $1 \leq s_p < t_p \leq l_p$, then

$$\begin{aligned} & \sum_{j=1}^{l_p} \left(\sum_{k=1}^j \frac{v_{j,p} u_{k,p}}{q_p} EA_{k,p} + \sum_{k=j+1}^{l_p} \frac{v_{k,p} u_{j,p}}{q_p} EA_{k,p} \right) EA_{j,p} \\ &= \left(\frac{v_{s_p} u_{s_p}}{q_p} EA_{s_p} + \frac{v_{t_p} u_{s_p}}{q_p} EA_{t_p} \right) (EA_{s_p}) + \left(\frac{v_{t_p} u_{s_p}}{q_p} EA_{s_p} + \frac{v_{t_p} u_{t_p}}{q_p} EA_{t_p} \right) (EA_{t_p}), \end{aligned}$$

so the equality follows from (2). \square

Let

$$L = L_S := \text{rank}(R)$$

be the number of the irreducible exceptional curves of $f : S' \rightarrow S$. We have

$$b_2(S') = 1 + L.$$

Note that S' has $H^1(S', \mathcal{O}_{S'}) = H^2(S', \mathcal{O}_{S'}) = 0$. Thus by Noether formula,

$$K_{S'}^2 = 12 - e(S') = 10 - b_2(S') = 9 - L.$$

Lemma 4.3. *Let S be a \mathbb{Q} -homology projective plane with cyclic singularities. Assume that K_S is not numerically trivial. Assume that S is not rational. If $L > 9$, then there is a (-1) -curve E on S' of the form (3.1) with $0 < m \leq \frac{\sqrt{D'}}{L-9}$.*

Proof. Since S is not rational and K_S is not numerically trivial, K_S is ample. Thus $m > 0$ for any (-1) -curve E by Lemma 3.7(5).

Since $K_{S'}^2 = 9 - L < 0$, S' is not a minimal surface. Let

$$g : S' = S_k \rightarrow S_{k-1} \rightarrow S_{k-2} \rightarrow \cdots \rightarrow S_1 \rightarrow S_0 = S_{\min}$$

be a morphism of S' to its minimal model. Since $K_{S_{\min}}^2 \geq 0$, we see that

$$k \geq L - 9.$$

Also one can write

$$K_{S'} = g^* K_{S_{\min}} + \sum_{i=1}^k E_i$$

where E_i is the total transform of the exceptional curve of the blowup $S_i \rightarrow S_{i-1}$. Note that E_1, \dots, E_k are effective divisors, not necessarily irreducible, satisfying $E_i^2 = -1$ and $E_i E_j = 0$ for $i \neq j$.

Let m_0 be the leading coefficient of $g^* K_{S_{\min}}$ written in the form (3.1). Since S is not rational, $K_{S_{\min}}$ is a nef \mathbb{Q} -divisor on S_{\min} , so $g^* K_{S_{\min}}$ is a nef \mathbb{Q} -divisor on S' . Since K_S is ample, this implies that $m_0 \geq 0$. Let m_i be the leading coefficient of E_i written in the form (3.1). Note that $\sqrt{D'}$ is the leading coefficient of $K_{S'}$ written in the form (3.1). Thus

$$\sqrt{D'} = m_0 + \sum_{i=1}^k m_i.$$

If E_s is a (-1) -curve and is a component of E_t for some $t \neq s$, then one can write $E_t = aE_s + F$ where $a \geq 1$ is an integer and F is an effective divisor. It follows that $m_t \geq am_s \geq m_s$. Let

$$m := \min\{m_1, m_2, \dots, m_k\}.$$

Then there is an irreducible member E among E_1, \dots, E_k whose leading coefficient is m . It is a (-1) -curve, and

$$\sqrt{D'} = m_0 + \sum_{i=1}^k m_i \geq \sum_{i=1}^k m_i \geq km \geq (L-9)m.$$

□

5. FIRST REDUCTION STEPS FOR THE CASES WITH $|Sing(S)| \geq 4$

Let S be a \mathbb{Q} -homology projective plane with cyclic quotient singularities such that $H_1(S^0, \mathbb{Z}) = 0$. By Lemma 3.6(3), the orders of singularities are pairwise relatively prime. Since $e_{orb}(S) \geq 0$ (Theorems 3.2 and 3.3), one can immediately see that S can have at most 4 singular points (also see [HK1], [Kol08]).

Assume that $|Sing(S)| = 4$. Then we enumerate all possible 4-tuples of orders of local fundamental groups:

- (1) $(2, 3, 5, q)$, $q \geq 7$, $\gcd(q, 30) = 1$,
- (2) $(2, 3, 7, q)$, $11 \leq q \leq 41$, $\gcd(q, 42) = 1$,
- (3) $(2, 3, 11, 13)$.

For (2) and (3), there are exactly 1092 different possible types for R , the sublattice of $H^2(S', \mathbb{Z})_{free}$ generated by all exceptional curves of the minimal resolution $f : S' \rightarrow S$. There are 2 types, $[3]$, $[2, 2]$, of order 3; 4 types, $[7]$, $[4, 2]$, $[3, 2, 2]$, A_6 , of order 7; $\frac{\phi(q)}{2} + 1$ types of order q , so the total number of types of R for the case $(2, 3, 7, q)$ is

$$2 \times 4 \times \left(\frac{\phi(q)}{2} + 1\right) = 4(\phi(q) + 2),$$

where ϕ is the Euler function. Here we identify $\frac{1}{q}(1, q_1)$ with $\frac{1}{q}(1, q_l)$. By Lemma 3.6(5), the number

$$D = |\det(R)|K_S^2$$

must be a nonzero square number. Among the 1092 cases, a computer calculation of the number D shows that only 24 cases satisfy this property. Table 1 describes these 24 cases.

The number D can be computed as follows. First note that

$$|\det(R)| = \text{the product of orders.}$$

To compute K_S^2 , we use the equality from 3.1,

$$K_S^2 = K_{S'}^2 + \sum_p \mathcal{D}_p K_{S'}.$$

By Noether formula,

$$K_{S'}^2 = 9 - L$$

where $L := \text{rank}(R)$ is the number of the exceptional curves of f . Finally the intersection number $\mathcal{D}_p K_{S'}$ is given in Lemma 3.1.

Remark 5.1. None of the 24 cases of Table 1 can be ruled out by any further lattice theoretic argument. In fact, in each case the lattice R can be embedded into a unimodular lattice $I_{1,L}(\text{odd})$ or $II_{1,L}(\text{even})$ of signature $(1, L)$. This can be checked by the local-global principle and the computation of ϵ -invariants (see e.g., [HK1] Section 6).

TABLE 1.

No.	Type of R	orders	K_S^2		$3e_{orb}(S)$
1	$[2] + A_2 + [7] + [13]$	$(2, 3, 7, 13)$	$\frac{1536}{91}$	$>$	$\frac{29}{182}$
2	$[2] + A_2 + [7] + [3, 2, 2, 2, 2, 2, 2, 2]$	$(2, 3, 7, 19)$	$\frac{6}{133}$	$<$	$\frac{23}{266}$
3	$[2] + A_2 + [7] + [5, 4]$	$(2, 3, 7, 19)$	$\frac{1350}{133}$	$>$	$\frac{23}{266}$
4	$[2] + A_2 + [7] + [3, 4, 2]$	$(2, 3, 7, 19)$	$\frac{1014}{133}$	$>$	$\frac{23}{266}$
5	$[2] + A_2 + [4, 2] + [2, 2, 4, 2, 2, 2]$	$(2, 3, 7, 31)$	$\frac{150}{217}$	$>$	$\frac{11}{434}$
6	$[2] + A_2 + [4, 2] + [6, 2, 2, 2, 2, 2]$	$(2, 3, 7, 31)$	$\frac{486}{217}$	$>$	$\frac{11}{434}$
7	$[2] + [3] + [3, 2, 2] + [4, 2, 2, 2, 3]$	$(2, 3, 7, 29)$	$\frac{968}{609}$	$>$	$\frac{13}{406}$
8	$[2] + A_2 + [3, 2, 2] + [7, 2, 2, 2]$	$(2, 3, 7, 25)$	$\frac{24}{7}$	$>$	$\frac{17}{350}$
9	$[2] + A_2 + [7] + [2, 2, 3, 2, 2, 2, 2, 2, 2]$	$(2, 3, 7, 31)$	$\frac{54}{217}$	$>$	$\frac{11}{434}$
10	$[2] + [3] + [4, 2] + [3, 3, 2, 2, 3]$	$(2, 3, 7, 41)$	$\frac{2888}{861}$	$>$	$\frac{1}{574}$
11	$[2] + A_2 + [3, 2, 2] + [7, 2, 2, 2, 2, 2]$	$(2, 3, 7, 37)$	$\frac{384}{259}$	$>$	$\frac{5}{518}$
12	$[2] + A_2 + [4, 2] + [11, 2, 2]$	$(2, 3, 7, 31)$	$\frac{2166}{217}$	$>$	$\frac{11}{434}$
13	$[2] + [3] + A_6 + [2, 6, 2, 2]$	$(2, 3, 7, 29)$	$\frac{56}{87}$	$>$	$\frac{13}{406}$
14	$[2] + [3] + [3, 2, 2] + [4, 3]$	$(2, 3, 7, 11)$	$\frac{1058}{231}$	$>$	$\frac{31}{154}$
15	$[2] + [3] + [3, 2, 2] + [3, 2, 2, 2, 2]$	$(2, 3, 7, 11)$	$\frac{50}{231}$	$>$	$\frac{31}{154}$
16	$[2] + [3] + [3, 2, 2] + [4, 2, 2, 3]$	$(2, 3, 7, 23)$	$\frac{1250}{483}$	$>$	$\frac{19}{322}$
17	$[2] + [3] + [3, 2, 2] + [6, 5]$	$(2, 3, 7, 29)$	$\frac{5000}{609}$	$>$	$\frac{13}{406}$
18	$[2] + A_2 + [3, 2, 2] + [3, 5, 2]$	$(2, 3, 7, 25)$	$\frac{24}{7}$	$>$	$\frac{17}{350}$
19	$[2] + A_2 + [3, 2, 2] + [13, 2]$	$(2, 3, 7, 25)$	$\frac{1944}{175}$	$>$	$\frac{17}{350}$
20	$[2] + A_2 + [4, 2] + [4, 2, 2, 2]$	$(2, 3, 7, 13)$	$\frac{216}{91}$	$>$	$\frac{29}{182}$
21	$[2] + A_2 + [4, 2] + [5, 2, 2]$	$(2, 3, 7, 13)$	$\frac{384}{91}$	$>$	$\frac{29}{182}$
22	$[2] + A_2 + [4, 2] + [4, 2, 2, 2, 2, 2]$	$(2, 3, 7, 19)$	$\frac{54}{133}$	$>$	$\frac{23}{266}$
23	$[2] + [3] + [3, 2, 2, 2, 2] + [4, 2, 2, 2]$	$(2, 3, 11, 13)$	$\frac{8}{429}$	$>$	$\frac{1}{286}$
24	$[2] + [3] + [3, 2, 2, 2, 2] + [5, 2, 2]$	$(2, 3, 11, 13)$	$\frac{800}{429}$	$>$	$\frac{1}{286}$

Lemma 5.2. *In all cases of Table 1 except the second case, $-K_S$ is ample. In the second case, S is rational.*

Proof. The 23 cases do not satisfy the inequality $K_S^2 \leq 3e_{orb}(S)$ in Theorem 3.2. Thus the first assertion follows.

Consider the second case $A_1 + A_2 + [7] + [3, 2, 2, 2, 2, 2, 2, 2]$. In this case,

$$K_S^2 = \frac{6}{133}, \quad D = |\det(R)|K_S^2 = 36, \quad L = 13.$$

Suppose that S is not rational. By Lemma 4.3, S' contains a (-1) -curve E with $0 < m \leq \frac{\sqrt{D}}{L-9} = \frac{6}{4}$, i.e., $m = 1$. By Proposition 4.2(1), we obtain

$$\sum_p \sum_j \left(1 - \frac{v_{j,p} + u_{j,p}}{q_p}\right) (EA_{j,p}) = -EK_{S'} + \frac{m}{\sqrt{D}} K_S^2 = 1 + \frac{1}{6} \cdot \frac{6}{133} = \frac{134}{133}.$$

Looking at Table 2, we see that there are non-negative integers x, y such that

TABLE 2.

	[2]	[2, 2]	[7]	[3, 2, 2, 2, 2, 2, 2, 2, 2]									
j	1	1	2	1	1	2	3	4	5	6	7	8	9
$1 - \frac{v_j + u_j}{q}$	0	0	0	$\frac{5}{7}$	$\frac{9}{19}$	$\frac{8}{19}$	$\frac{7}{19}$	$\frac{6}{19}$	$\frac{5}{19}$	$\frac{4}{19}$	$\frac{3}{19}$	$\frac{2}{19}$	$\frac{1}{19}$

$$\frac{5x}{7} + \frac{y}{19} = \frac{134}{133}.$$

But it is easy to check that this equation has no solution. \square

Next we consider the cases: $(2, 3, 5, q)$, $q \geq 7$, $\gcd(q, 30) = 1$.

Lemma 5.3. *In the cases $(2, 3, 5, q)$, $q \geq 7$, $\gcd(q, 30) = 1$, the order 3 singularity must be of type $\frac{1}{3}(1, 1)$.*

Proof. Suppose that it is of type A_2 . We divide the proof into 3 cases according to the type of the third singularity.

Case 1: $A_1 + A_2 + A_4 + \frac{1}{q}(1, q_1)$. In this case

$$K_S^2 = \sum_{j=1}^l n_j - 3l + \frac{q_1 + q_l + 2}{q},$$

and

$$D = 30\{q_1 + q_l + (\sum_{j=1}^l n_j - 3l)q + 2\}.$$

Since D is a square number, 3 divides $q_1 + q_l + (tr - 3l)q + 2 \equiv q_1 + q_l + (tr)q + 2$. Then, by Proposition 2.2, q is a multiple of 3, a contradiction.

Case 2: $A_1 + A_2 + \frac{1}{5}(1, 2) + \frac{1}{q}(1, q_1)$. In this case

$$K_S^2 = \sum_{j=1}^l n_j - 3l + \frac{12}{5} + \frac{q_1 + q_l + 2}{q},$$

and

$$D = 6[5(q_1 + q_l) + \{5(\sum_{j=1}^l n_j - 3l) + 12\}q + 10].$$

Thus 3 divides $5(q_1 + q_l) + \{5(tr - 3l) + 12\}q + 10 \equiv -(q_1 + q_l) - (tr)q + 1$. Then, by Proposition 2.2, q is a multiple of 3, a contradiction.

Case 3: $A_1 + A_2 + \frac{1}{5}(1, 1) + \frac{1}{q}(1, q_1)$. In this case

$$K_S^2 = \sum_{j=1}^l n_j - 3l + \frac{24}{5} + \frac{q_1 + q_l + 2}{q},$$

and

$$D = 6[5(q_1 + q_l) + \{5(\sum_{j=1}^l n_j - 3l) + 24\}q + 10].$$

Thus 3 divides $5(q_1 + q_l) + \{5(tr - 3l) + 24\}q + 10$. Then, by Proposition 2.2, q is a multiple of 3, a contradiction. \square

In the following two lemmas, we do not assume that $H_1(S^0, \mathbb{Z}) = 0$. So the orders may not be pairwise relatively prime.

Lemma 5.4. *Let S be a \mathbb{Q} -homology projective plane with exactly 4 cyclic singular points p_1, p_2, p_3, p_4 of orders $(2, 3, 5, q)$, $q \geq 7$. (We do not assume that $\gcd(q, 30) = 1$.) Regard $\mathcal{F} := f^{-1}(\text{Sing}(S))$ as a reduced integral divisor on S' . Assume that S' contains a (-1) -curve E . Then,*

$$E \cdot \mathcal{F} \geq 2.$$

The equality holds if and only if $E \cdot f^{-1}(p_i) = 0$ for $i = 1, 2, 3$ and $E \cdot f^{-1}(p_4) = 2$.

Proof. Assume that $E \cdot \mathcal{F} = 1$. Blowing up the intersection point, then contracting the proper transform of E and the proper transforms of all irreducible components of \mathcal{F} , we obtain a \mathbb{Q} -homology projective plane \bar{S} with 5 quotient singular points. Then, by [HK1], the minimal resolution of \bar{S} is an Enriques surface, hence has no (-1) -curve, which is a contradiction. This proves that $E \cdot \mathcal{F} \geq 2$.

Assume that $E \cdot \mathcal{F} = 2$.

Suppose that E meets an end component F of $f^{-1}(p_i)$ for some $1 \leq i \leq 3$.

If $EF = 1$, then $EF' = 1$ for some other component F' of $f^{-1}(p_j)$, where j may or may not be i . Assume that $E \cap F \cap F' = \emptyset$. Blowing up the intersection point of E and F' sufficiently many times, then contracting the proper transform of E with a string of (-2) -curves and the proper transforms of all irreducible components of \mathcal{F} , we obtain a \mathbb{Q} -homology projective plane \tilde{S} with 4 quotient singular points such that $e_{orb} < 0$ (see Lemma 2.4(6)), which violates the orbifold Bogomolov-Miyaoka-Yau inequality. Assume that $E \cap F \cap F' \neq \emptyset$. Blowing up the intersection point once, then contracting the proper transform of E and the proper transforms of all irreducible components of \mathcal{F} , we obtain a \mathbb{Q} -homology projective plane \tilde{S} with 6 quotient singular points, a contradiction to [HK1].

If E intersects F at 2 distinct points, then we get a similar contradiction: blowing up one of the two intersection points of E and F sufficiently many times, then contracting the proper transform of E with the adjacent string of (-2) -curves and the proper transforms of all irreducible components of \mathcal{F} , to obtain a \mathbb{Q} -homology projective plane \tilde{S} with 4 quotient singular points such that $e_{orb} < 0$.

If E intersects F at 1 point with multiplicity 2, then blowing up the intersection point twice and then contracting the proper transform of E with a (-2) -curve and the proper transforms of all irreducible components of \mathcal{F} , we obtain a \mathbb{Q} -homology projective plane \tilde{S} with 6 quotient singular points, a contradiction to [HK1].

We have proved that E does not meet any end component of $f^{-1}(p_i)$ for $1 \leq i \leq 3$. This implies that $E \cdot f^{-1}(p_1) = E \cdot f^{-1}(p_2) = 0$ and $E \cdot f^{-1}(p_3) = 0$ if $f^{-1}(p_3)$

has at most 2 components. We will show that $E \cdot f^{-1}(p_3) = 0$ even if $f^{-1}(p_3)$ has more than 2 components, i.e., p_3 is of type A_4 . Suppose that p_3 is of type A_4 and F_1, F_2, F_3, F_4 be its 4 components whose dual graph is $F_1 - F_2 - F_3 - F_4$.

If E meets F_2 at two distinct points, then blowing up one of the two intersection points of E and F_2 once, then contracting the proper transform of E and the proper transforms of all irreducible components of \mathcal{F} , we obtain a \mathbb{Q} -homology projective plane \bar{S} with one noncyclic quotient singularity of type

$$< 3; 2, 1; 2, 1; 3, 2 > := \begin{array}{c} \begin{array}{cccc} \circ^{-2} & - & \circ^{-3} & - & \circ^{-2} & - & \circ^{-2} \end{array} \\ | \\ \circ^{-2} \end{array}$$

and 3 cyclic singular points of order 2, 3, q (see [Br] or Table 1 of [HK1] for the notation of dual graphs of noncyclic singularities). This surface has

$$e_{orb} = -1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{q} + \frac{1}{48} < 0,$$

which violates the orbifold Bogomolov-Miyaoka-Yau inequality.

If $EF_2 = EF_3 = 1$ and $E \cap F_2 \cap F_3 = \emptyset$, then blowing up the intersection point of E and F_3 once, then contracting the proper transform of E and the proper transforms of all irreducible components of \mathcal{F} , we obtain a \mathbb{Q} -homology projective plane \bar{S} with one noncyclic quotient singularity of type

$$< 2; 2, 1; 2, 1; 5, 2 > := \begin{array}{c} \begin{array}{cccc} \circ^{-2} & - & \circ^{-2} & - & \circ^{-3} & - & \circ^{-2} \end{array} \\ | \\ \circ^{-2} \end{array}$$

and 3 cyclic singular points of order 2, 3, q . This surface has

$$e_{orb} = -1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{q} + \frac{1}{60} < 0,$$

which also violates the orbifold Bogomolov-Miyaoka-Yau inequality.

If $EF_2 = EF_3 = 1$ and $E \cap F_2 \cap F_3 \neq \emptyset$, then blowing up the intersection point once, then contracting the proper transform of E and the proper transforms of all irreducible components of \mathcal{F} , we obtain a \mathbb{Q} -homology projective plane \bar{S} with 6 quotient singular points, a contradiction to [HK1].

If $EF_2 = 1$ and $EF = 1$ for some component F of $f^{-1}(p_i)$ for some $i \neq 3$, then blowing up the intersection point of E and F four times, then contracting the proper transform of E with a string of three (-2) -curves and the proper transforms of all irreducible components of \mathcal{F} , we obtain a \mathbb{Q} -homology projective plane \bar{S} with one noncyclic quotient singularity of type $E_8 = < 2; 2, 1; 3, 2; 5, 4 >$

$$< 2; 2, 1; 3, 2; 5, 4 > := \begin{array}{c} \begin{array}{ccccccc} \circ^{-2} & - & \circ^{-2} & - & \circ^{-2} & - & \circ^{-2} & - & \circ^{-2} & - & \circ^{-2} & - & \circ^{-2} \end{array} \\ | \\ \circ^{-2} \end{array}$$

and 3 cyclic singular points of order $\geq 2, \geq 3, \geq q$. This surface has

$$e_{orb} \leq -1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{q} + \frac{1}{120} < 0,$$

which violates the orbifold Bogomolov-Miyaoka-Yau inequality.

This completes the proof of $E.f^{-1}(p_3) = 0$. Thus $E.f^{-1}(p_4) = 2$. \square

In the following lemma, we do not assume that $H_1(S^0, \mathbb{Z}) = 0$.

Lemma 5.5. *Let S be a \mathbb{Q} -homology projective plane with exactly 4 cyclic singular points p_1, p_2, p_3, p_4 of orders $(2, 3, 5, q)$. (We do not assume that $\gcd(q, 30) = 1$.) Assume that K_S is ample. Assume that the order 3 singularity is of type $\frac{1}{3}(1, 1)$. Then the following hold true.*

- (1) $L \geq 12$ except possibly four cases, No.1 – 4 in Table 3. In each of these four cases, S is rational and $L = 11$.
- (2) $q \geq 20$ except possibly one case, No.1 in Table 3.

Proof. (1) We have to consider the following types.

- $A_1 + \frac{1}{3}(1, 1) + A_4 + \frac{1}{q}(1, q_1)$
- $A_1 + \frac{1}{3}(1, 1) + \frac{1}{5}(1, 2) + \frac{1}{q}(1, q_1)$
- $A_1 + \frac{1}{3}(1, 1) + \frac{1}{5}(1, 1) + \frac{1}{q}(1, q_1)$

Let $[n_1, \dots, n_l]$ be the Hirzebruch-Jung continued fraction corresponding to the singularity p_4 . Since K_S is ample, Theorem 3.2 implies that

$$0 < K_{S'}^2 - \mathcal{D}_{p_2}^2 - \mathcal{D}_{p_3}^2 - \mathcal{D}_{p_4}^2 = K_S^2 \leq 3e_{orb}(S) = \frac{1}{10} + \frac{3}{q}.$$

Since $K_{S'}^2 = 9 - L$, $\mathcal{D}_{p_2}^2 = -\frac{1}{3}$, Lemma 3.1 implies that

$$L - 7 + 2l - \frac{1}{3} + \mathcal{D}_{p_3}^2 - \frac{q_1 + q_l + 2}{q} < \sum n_j \leq L - 7 + 2l - \frac{1}{3} + \mathcal{D}_{p_3}^2 - \frac{q_1 + q_l - 1}{q} + \frac{1}{10}.$$

In particular, if L is bounded, so is the number of possible cases for $[n_1, \dots, n_l]$.

Assume that $L \leq 11$.

If p_3 is of type A_4 , then $L = l + 6$, $\mathcal{D}_{p_3}^2 = 0$ and the above inequality shows that $\sum n_j = 3l - 2$ or $3l - 3$, so up to permutation of n_1, \dots, n_l ,

$$\begin{aligned} [n_1, \dots, n_l] &= [5, 2, 2, 2, 2], [4, 3, 2, 2, 2], [3, 3, 3, 2, 2]; \\ &\quad [4, 2, 2, 2, 2], [3, 3, 2, 2, 2]; \\ &\quad [4, 2, 2, 2], [3, 3, 2, 2]; \\ &\quad [3, 2, 2, 2]; \\ &\quad [3, 2, 2]; \\ &\quad [2, 2, 2]; \\ &\quad [2, 2], \end{aligned}$$

hence there are 42 possible cases for $[n_1, \dots, n_l]$. Here we identify $[n_1, \dots, n_l]$ with its reverse $[n_l, \dots, n_1]$.

If p_3 is of type $\frac{1}{5}(1, 2)$, then $L = l + 4$, $\mathcal{D}_{p_3}^2 = -\frac{2}{5}$ and $\sum n_j = 3l - 4$ or $3l - 5$, so up to permutation of n_1, \dots, n_l ,

$$\begin{aligned} [n_1, \dots, n_l] &= [5, 2, 2, 2, 2, 2, 2], [4, 3, 2, 2, 2, 2, 2], [3, 3, 3, 2, 2, 2, 2]; \\ &\quad [4, 2, 2, 2, 2, 2, 2], [3, 3, 2, 2, 2, 2, 2]; \\ &\quad [4, 2, 2, 2, 2, 2], [3, 3, 2, 2, 2, 2]; \\ &\quad [3, 2, 2, 2, 2, 2]; \\ &\quad [3, 2, 2, 2, 2]; \\ &\quad [2, 2, 2, 2, 2]; \\ &\quad [2, 2, 2, 2], \end{aligned}$$

hence there are 80 possible cases for $[n_1, \dots, n_l]$ if $l \leq 7$.

If p_3 is of type $\frac{1}{5}(1, 1)$, then $L = l + 3$, $\mathcal{D}_{p_3}^2 = -\frac{9}{5}$ and $\sum n_j = 3l - 7$ or $3l - 8$, so up to permutation of n_1, \dots, n_l ,

$$[n_1, \dots, n_l] = [3, 2, 2, 2, 2, 2, 2, 2], [2, 2, 2, 2, 2, 2, 2, 2], [2, 2, 2, 2, 2, 2, 2],$$

hence there are 6 possible cases for $[n_1, \dots, n_l]$ if $l \leq 8$.

Among these $42 + 80 + 6 = 128$ cases, a direct calculation of $D = |\det(R)|K_S^2$ shows that only 11 cases satisfy the condition that D is a positive square number (see Lemma 3.6(5)). Table 3 describes the 11 cases.

TABLE 3.

No.	Type of R	q	K_S^2		$3e_{orb}$
1	$A_1 + \frac{1}{3}(1, 1) + \frac{1}{5}(1, 1) + [2, 2, 2, 2, 2, 2, 2]$	9	$\frac{2}{15}$	$<$	$\frac{13}{30}$
2	$A_1 + \frac{1}{3}(1, 1) + \frac{1}{5}(1, 2) + [4, 2, 2, 2, 2, 2, 2]$	22	$\frac{1}{165}$	$<$	$\frac{13}{55}$
3	$A_1 + \frac{1}{3}(1, 1) + \frac{1}{5}(1, 2) + [3, 3, 2, 2, 2, 2, 2]$	33	$\frac{2}{55}$	$<$	$\frac{21}{110}$
4	$A_1 + \frac{1}{3}(1, 1) + \frac{1}{5}(1, 2) + [3, 2, 2, 3, 2, 2, 2]$	43	$\frac{8}{645}$	$<$	$\frac{73}{430}$
5	$A_1 + \frac{1}{3}(1, 1) + \frac{1}{5}(1, 2) + [2, 2, 2, 4, 2, 2, 2]$	40	$\frac{1}{3}$	$>$	$\frac{7}{40}$
6	$A_1 + \frac{1}{3}(1, 1) + \frac{1}{5}(1, 2) + [3, 3, 3, 2, 2, 2, 2]$	73	$\frac{1058}{1095}$	$>$	$\frac{103}{730}$
7	$A_1 + \frac{1}{3}(1, 1) + \frac{1}{5}(1, 2) + [2, 3, 4, 2, 2, 2, 2]$	70	$\frac{25}{21}$	$>$	$\frac{1}{7}$
8	$A_1 + \frac{1}{3}(1, 1) + \frac{1}{5}(1, 2) + [2, 3, 3, 3, 2, 2, 2]$	97	$\frac{1682}{1455}$	$>$	$\frac{127}{970}$
9	$A_1 + \frac{1}{3}(1, 1) + \frac{1}{5}(1, 2) + [2, 2, 4, 3, 2, 2, 2]$	78	$\frac{81}{65}$	$>$	$\frac{9}{65}$
10	$A_1 + \frac{1}{3}(1, 1) + \frac{1}{5}(1, 2) + [3, 3, 2, 2, 3, 2, 2]$	87	$\frac{128}{145}$	$>$	$\frac{39}{290}$
11	$A_1 + \frac{1}{3}(1, 1) + \frac{1}{5}(1, 2) + [2, 3, 3, 2, 2, 3, 2]$	103	$\frac{1568}{1545}$	$>$	$\frac{133}{1030}$

Among the 11 cases, only the first 4 cases satisfy the orbifold Bogomolov-Miyaoka-Yau inequality $K_S^2 \leq 3e_{orb}$.

As for the first 4 cases of Table 3, one can check that none of them can be ruled out by any further lattice theoretic argument, i.e., in each case the lattice R can be embedded into an odd unimodular lattice of signature $(1, L)$. This can be checked by the local-global principle and the computation of ϵ -invariants (see e.g., [HK1] Section 6).

To prove the rationality in each of the first 4 cases of Table 3, we will use the formulae from Proposition 4.2. First note that $L = 11$ in each of the first 4 cases of Table 3.

Case 1. Suppose that this case occurs on S which is not rational. Note that $D = 36$. Since $\text{disc}(\bar{R})$ is a cyclic group (Lemma 3.7), we see that $\det(\bar{R}) = \frac{\det(R)}{3^2}$, and hence $D' = \frac{D}{3^2} = 4$. By Lemma 4.3, S' contains a (-1) -curve E with $0 < m \leq \frac{\sqrt{D'}}{L-9} = 1$, i.e., $m = 1$. By Proposition 4.2(1), we obtain

$$\sum_p \sum_j \left(1 - \frac{v_{j,p} + u_{j,p}}{q_p}\right) (EA_{j,p}) = 1 + \frac{m}{\sqrt{D'}} K_S^2 = \frac{16}{15}.$$

Looking at Table 4, we see that there are non-negative integers x, y such that

$$\frac{x}{3} + \frac{3y}{5} = \frac{16}{15}.$$

It is easy to check that the equation has no solution.

TABLE 4.

	[2]	[3]	[5]	[2, 2, 2, 2, 2, 2, 2, 2]							
j	1	1	1	1	2	3	4	5	6	7	8
$1 - \frac{v_j + u_j}{q}$	0	$\frac{1}{3}$	$\frac{3}{5}$	0	0	0	0	0	0	0	0

Case 2. Suppose that this case occurs on S which is not rational. Note that $D = 4$. Since $\text{disc}(\bar{R})$ is a cyclic group (Lemma 3.7), we see that $D' = \frac{D}{2^2} = 1$. By Lemma 4.3, S' contains a (-1) -curve E with $0 < m \leq \frac{\sqrt{D'}}{L-9} = \frac{1}{2}$, a contradiction.

Case 3. Suppose that this case occurs on S which is not rational. Note that $D = 36$. Since $\text{disc}(\bar{R})$ is a cyclic group (Lemma 3.7), we see that $D' = \frac{D}{3^2} = 4$. By Lemma 4.3, S' contains a (-1) -curve E with $0 < m \leq \frac{\sqrt{D'}}{L-9} = 1$, i.e., $m = 1$. By Proposition 4.2 (1), we obtain

$$\sum_p \sum_j \left(1 - \frac{v_{j,p} + u_{j,p}}{q_p}\right) (EA_{j,p}) = 1 + \frac{m}{\sqrt{D'}} K_S^2 = \frac{56}{55}.$$

Looking at Table 5, we see that there are non-negative integers x, y, z such that

TABLE 5.

	[2]	[3]	[2, 3]		[3, 3, 2, 2, 2, 2, 2]						
j	1	1	1	2	1	2	3	4	5	6	7
$1 - \frac{v_j + u_j}{q}$	0	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{19}{33}$	$\frac{24}{33}$	$\frac{20}{33}$	$\frac{16}{33}$	$\frac{12}{33}$	$\frac{8}{33}$	$\frac{4}{33}$
$\frac{v_j u_j}{q}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{13}{33}$	$\frac{18}{33}$	$\frac{40}{33}$	$\frac{52}{33}$	$\frac{54}{33}$	$\frac{46}{33}$	$\frac{28}{33}$

$$\frac{x}{3} + \frac{y}{5} + \frac{z}{33} = \frac{56}{55}.$$

The equation has 3 solutions $(x, y, z) = (0, 1, 27), (1, 1, 16), (2, 1, 5)$. Again by Table 5, we can rule out the third solution. By Proposition 4.2(2), we obtain

$$\sum_p \sum_j \frac{v_j u_j}{q} (EA_j)^2 \leq 1 + \frac{m^2}{D'} K_S^2 = \frac{111}{110},$$

which rules out the first two solutions.

Case 4. Suppose that this case occurs on S which is not rational. Note that $D = 4^2$. Since the orders are pairwise relatively prime, $D' = D$. By

Lemma 4.3, S' contains a (-1) -curve E with $0 < m \leq \frac{\sqrt{D}}{L-9} = 2$, i.e., $m = 1$ or 2 . By Proposition 4.2, we obtain

$$\sum_p \sum_j \left(1 - \frac{v_{j,p} + u_{j,p}}{q_p}\right) (EA_{j,p}) = 1 + \frac{m}{\sqrt{D}} K_S^2 = \frac{647}{645} \quad \text{or} \quad \frac{649}{645}.$$

Looking at Table 6, we see that there are non-negative integers x, y, z such that

TABLE 6.

	[2]	[3]	[2, 3]		[3, 2, 2, 3, 2, 2, 2]						
j	1	1	1	2	1	2	3	4	5	6	7
$1 - \frac{v_j + u_j}{q}$	0	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{23}{43}$	$\frac{26}{43}$	$\frac{29}{43}$	$\frac{32}{43}$	$\frac{24}{43}$	$\frac{16}{43}$	$\frac{8}{43}$

$$\frac{x}{3} + \frac{y}{5} + \frac{z}{43} = \frac{647}{645} \quad \text{or} \quad \frac{649}{645}.$$

But it is easy to check that both equations have no solution.

(2) Suppose $2 \leq q \leq 19$. A direct calculation shows that only 6 cases satisfy the condition that D is a positive square number. Table 7 contains the 6 cases. Among the 6 cases, only the first case satisfies the orbifold Bogomolov-Miyaoka-Yau

TABLE 7.

Type of R	q	K_S^2		$3e_{orb}$
$A_1 + \frac{1}{3}(1, 1) + \frac{1}{5}(1, 1) + A_8$	9	$\frac{2}{15}$	$<$	$\frac{13}{30}$
$A_1 + \frac{1}{3}(1, 1) + A_4 + \frac{1}{4}(1, 1)$	4	$\frac{10}{3}$	$>$	$\frac{17}{20}$
$A_1 + \frac{1}{3}(1, 1) + A_4 + \frac{1}{5}(1, 2)$	5	$\frac{26}{15}$	$>$	$\frac{7}{10}$
$A_1 + \frac{1}{3}(1, 1) + A_4 + \frac{1}{6}(1, 1)$	6	5	$>$	$\frac{3}{5}$
$A_1 + \frac{1}{3}(1, 1) + \frac{1}{5}(1, 1) + \frac{1}{6}(1, 1)$	6	$\frac{49}{5}$	$>$	$\frac{3}{5}$
$A_1 + \frac{1}{3}(1, 1) + A_4 + \frac{1}{16}(1, 3)$	16	$\frac{10}{3}$	$>$	$\frac{23}{80}$

inequality $K_S^2 \leq 3e_{orb}$. But it is already considered in (1). \square

Lemma 5.6. *Let S be a \mathbb{Q} -homology projective plane with exactly 4 cyclic singular points p_1, p_2, p_3, p_4 of orders $(2, 3, 7, q)$, $11 \leq q \leq 41$, or $(2, 3, 11, 13)$. Regard $\mathcal{F} := f^{-1}(\text{Sing}(S))$ as a reduced integral divisor on S' . Assume that S' contains a (-1) -curve E . Then,*

$$E \cdot \mathcal{F} \geq 2.$$

Moreover, if $E \cdot \mathcal{F} = 2$, then E does not meet an end component of $f^{-1}(p_i)$ for any $i = 1, 2, 3, 4$.

Proof. The proof of the first assertion is the same as that of Lemma 5.4.

To prove the second assertion, assume that $E \cdot \mathcal{F} = 2$.

Suppose that E meets an end component F of $f^{-1}(p_i)$ for some $1 \leq i \leq 4$.

If $EF = 1$, then $EF' = 1$ for some other component F' of $f^{-1}(p_j)$, where j may or may not be i . Assume that $E \cap F \cap F' = \emptyset$. Blowing up the intersection point of E and F' sufficiently many times, then contracting the proper transform of E with a string of (-2) -curves and the proper transforms of all irreducible components of \mathcal{F} , we obtain a \mathbb{Q} -homology projective plane \bar{S} with 4 quotient singular points such that $e_{orb} < 0$ (see Lemma 2.4(6)), which violates the orbifold Bogomolov-Miyaoka-Yau inequality. Assume that $E \cap F \cap F' \neq \emptyset$. Blowing up the intersection point once, then contracting the proper transform of E and the proper transforms of all irreducible components of \mathcal{F} , we obtain a \mathbb{Q} -homology projective plane \bar{S} with 6 quotient singular points, a contradiction to [HK1].

If E intersects F at 2 distinct points, then we get a similar contradiction: blowing up one of the two intersection points of E and F sufficiently many times, then contracting the proper transform of E with the adjacent string of (-2) -curves and the proper transforms of all irreducible components of \mathcal{F} , to obtain a \mathbb{Q} -homology projective plane \bar{S} with 4 quotient singular points such that $e_{orb} < 0$.

If E intersects F at 1 point with multiplicity 2, then blowing up the intersection point twice and then contracting the proper transform of E with a (-2) -curve and the proper transforms of all irreducible components of \mathcal{F} , we obtain a \mathbb{Q} -homology projective plane \bar{S} with 6 quotient singular points, a contradiction to [HK1].

In all cases, we get a contradiction. This proves the second assertion. \square

6. PROOF OF THEOREM 1.4

Let S be a \mathbb{Q} -homology projective plane with cyclic quotient singularities such that

- $H_1(S^0, \mathbb{Z}) = 0$,
- S is not rational.

Assume that $|Sing(S)| = 4$. In the previous section, we have enumerated all possible 4-tuples of orders of local fundamental groups:

- (1) $(2, 3, 5, q)$, $q \geq 7$, $\gcd(q, 30) = 1$,
- (2) $(2, 3, 7, q)$, $11 \leq q \leq 41$, $\gcd(q, 42) = 1$,
- (3) $(2, 3, 11, 13)$.

For (2) and (3), we have seen that there are 24 different possible types for R , the sublattice of $H^2(S', \mathbb{Z})_{free}$ generated by all exceptional curves of the minimal resolution $f : S' \rightarrow S$, as shown in Table 1. Lemma 5.2 rules out all these 24 cases, since we assume that S is not rational.

For (1), the order 3 singularity is of type $\frac{1}{3}(1, 1)$ (Lemma 5.3), so it remains to consider the following cases:

- $A_1 + \frac{1}{3}(1, 1) + A_4 + \frac{1}{q}(1, q_1)$, $q \geq 7$, $\gcd(q, 30) = 1$;
- $A_1 + \frac{1}{3}(1, 1) + \frac{1}{5}(1, 2) + \frac{1}{q}(1, q_1)$, $q \geq 7$, $\gcd(q, 30) = 1$;
- $A_1 + \frac{1}{3}(1, 1) + \frac{1}{5}(1, 1) + \frac{1}{q}(1, q_1)$, $q \geq 7$, $\gcd(q, 30) = 1$.

Since S is not rational, K_S is ample by Lemma 3.6(4).

By Lemma 5.5 we may also assume that

- $q \geq 20$ and $L \geq 12$.

We will show that none of the above cases occurs. In the following proof we do not assume that $\gcd(q, 30) = 1$ (so do not assume that $H_1(S^0, \mathbb{Z}) = 0$), but still assume that K_S is ample. That is, in the following proof we will consider the cases

- $A_1 + \frac{1}{3}(1, 1) + A_4 + \frac{1}{q}(1, q_1)$, $q \geq 20$ and $L \geq 12$;
- $A_1 + \frac{1}{3}(1, 1) + \frac{1}{5}(1, 2) + \frac{1}{q}(1, q_1)$, $q \geq 20$ and $L \geq 12$;
- $A_1 + \frac{1}{3}(1, 1) + \frac{1}{5}(1, 1) + \frac{1}{q}(1, q_1)$, $q \geq 20$ and $L \geq 12$

with the assumption that

- K_S is ample and S is not rational.

We will show that none of these cases occurs. The reason why we consider the situation without the assumption that $\gcd(q, 30) = 1$ is that some part of the proof below uses induction on $L = \text{rank}(R)$. After blowing down a suitable (-1) -curve E on S' ,

$$S' \rightarrow S'',$$

we contract Hirzebruch-Jung chains of rational curves,

$$S'' \rightarrow \bar{S},$$

to get a new \mathbb{Q} -homology projective plane \bar{S} with $L_{\bar{S}} = L - 1$ having cyclic quotient singularities whose orders may not be pairwise relatively prime.

By Lemma 4.3, there is a (-1) -curve E on S' of the form (3.1) with

$$0 < \frac{m}{\sqrt{D'}} \leq \frac{1}{L-9} \leq \frac{1}{3}.$$

We will show that the existence of such a curve E leads to a contradiction.

Step 1.

- (1) $K_S^2 \leq \frac{1}{4}$,
- (2) $\frac{m}{\sqrt{D'}} K_S^2 \leq \frac{1}{12}$,
- (3) $\frac{m^2}{D'} K_S^2 \leq \frac{1}{36}$.

Proof. Since $q \geq 20$,

$$3e_{orb}(S) = \frac{1}{10} + \frac{3}{q} \leq \frac{1}{10} + \frac{3}{20} = \frac{1}{4}.$$

Since K_S is ample, (1) follows from the orbifold BMY inequality. (2) and (3) follow from (1) and the inequality $\frac{m}{\sqrt{D'}} \leq \frac{1}{3}$. \square

Let p_1, p_2, p_3, p_4 be the 4 singular points. Assume that the singularity p_4 is of type $[n_1, \dots, n_l]$. Since $L \geq 12$, we see that $l \geq 6$.

Step 2. $E.f^{-1}(p_4) = 2$ and $E.f^{-1}(p_i) = 0$ for $i = 1, 2, 3$.

Proof. By Proposition 4.2(1)

$$\sum_p \sum_{j=1}^{l_p} \left(1 - \frac{v_{j,p} + u_{j,p}}{q_p}\right) (EA_{j,p}) = 1 + \frac{m}{\sqrt{D'}} K_S^2.$$

By Lemma 2.4 we see that $1 - \frac{v_{j,p} + u_{j,p}}{q_p} \geq 0$ for all j, p , so by looking at only the terms with $p = p_4$, we get

$$\begin{aligned} E.f^{-1}(p_4) - \sum_{j=1}^l \left(\frac{v_j + u_j}{q} \right) (EA_j) &= \sum_{j=1}^l \left(1 - \frac{v_j + u_j}{q} \right) (EA_j) \\ &\leq 1 + \frac{m}{\sqrt{D'}} K_S^2, \end{aligned}$$

where $A_j := A_{j,p_4}$, $v_j := v_{j,p_4}$, $u_j := u_{j,p_4}$. By Proposition 4.2(2)

$$\sum_{j=1}^l \frac{v_j u_j}{q} (EA_j)^2 \leq 1 + \frac{m^2}{D'} K_S^2.$$

Adding these two inequalities side by side, we get

$$E.f^{-1}(p_4) - \sum_{j=1}^l \left(\frac{v_j + u_j}{q} \right) (EA_j) + \sum_{j=1}^l \frac{v_j u_j}{q} (EA_j)^2 \leq 2 + \frac{m}{\sqrt{D'}} K_S^2 + \frac{m^2}{D'} K_S^2.$$

By Lemma 2.5,

$$\sum_{j=1}^l \left(\frac{v_j + u_j}{q} \right) (EA_j) \leq \sum_{j=1}^l \frac{v_j u_j}{q} (EA_j)^2 + \frac{2}{q}.$$

Thus

$$E.f^{-1}(p_4) \leq 2 + \frac{m}{\sqrt{D'}} K_S^2 + \frac{m^2}{D'} K_S^2 + \frac{2}{q} < 3,$$

proving that $E.f^{-1}(p_4) \leq 2$.

Assume that $E.f^{-1}(p_4) = 2$. By Proposition 4.2(1),(2)

$$\sum_{p \neq p_4} \sum_{j=1}^{l_p} \left(1 - \frac{v_{j,p} + u_{j,p}}{q_p} \right) (EA_{j,p}) = 1 + \frac{m}{\sqrt{D'}} K_S^2 - E.f^{-1}(p_4) + \sum_{j=1}^l \left(\frac{v_j + u_j}{q} \right) (EA_j),$$

$$\sum_{p \neq p_4} \sum_{j=1}^{l_p} \frac{v_{j,p} u_{j,p}}{q_p} (EA_{j,p})^2 \leq 1 + \frac{m^2}{D'} K_S^2 - \sum_{j=1}^l \frac{v_j u_j}{q} (EA_j)^2.$$

Adding these two side by side, then using Lemma 2.5, we get

$$\begin{aligned} &\sum_{p \neq p_4} \sum_{j=1}^{l_p} \left(\left(1 - \frac{v_{j,p} + u_{j,p}}{q_p} \right) (EA_{j,p}) + \frac{v_{j,p} u_{j,p}}{q_p} (EA_{j,p})^2 \right) \\ &\leq \frac{m}{\sqrt{D'}} K_S^2 + \frac{m^2}{D'} K_S^2 + \sum_{j=1}^l \left(\frac{v_j + u_j}{q} \right) (EA_j) - \sum_{j=1}^l \frac{v_j u_j}{q} (EA_j)^2 \\ &\leq \frac{m}{\sqrt{D'}} K_S^2 + \frac{m^2}{D'} K_S^2 + \frac{2}{q} \\ &\leq \frac{1}{12} + \frac{1}{36} + \frac{2}{20} < \frac{1}{3}. \end{aligned}$$

Now from Table 8 it is easy to see that $E.f^{-1}(p_i) = 0$ for $i = 1, 2, 3$.

Assume that $E.f^{-1}(p_4) = 1$, i.e., $EA_s = 1$ for some s and $EA_j = 0$ for all $j \neq s$. Lemma 2.5 gives

$$\sum_{j=1}^l \left(\frac{v_j + u_j}{q} \right) (EA_j) \leq \sum_{j=1}^l \frac{v_j u_j}{q} (EA_j)^2 + \frac{1}{q}.$$

TABLE 8.

	[2]	[3]	[5]	[3, 2]		[2, 2, 2, 2]			
j	1	1	1	1	2	1	2	3	4
$1 - \frac{v_j+u_j}{q}$	0	$\frac{1}{3}$	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	0	0	0	0
$\frac{v_j u_j}{q}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{6}{5}$	$\frac{6}{5}$	$\frac{4}{5}$

Thus

$$\begin{aligned}
& \sum_{p \neq p_4} \sum_{j=1}^{l_p} \left(\left(1 - \frac{v_{j,p} + u_{j,p}}{q_p}\right) (EA_{j,p}) + \frac{v_{j,p} u_{j,p}}{q_p} (EA_{j,p})^2 \right) \\
& \leq 1 + \frac{m}{\sqrt{D'}} K_S^2 + \frac{m^2}{D'} K_S^2 + \frac{1}{q} \\
& \leq 1 + \frac{1}{12} + \frac{1}{36} + \frac{1}{20} < \frac{7}{6}.
\end{aligned}$$

Now Table 8 easily gives $E \cdot (f^{-1}(p_1) + f^{-1}(p_2) + f^{-1}(p_3)) \leq 1$. But this contradicts Lemma 5.4.

Assume that $E \cdot f^{-1}(p_4) = 0$.

In this case, we have

$$\sum_{p \neq p_4} \sum_{j=1}^{l_p} \left(1 - \frac{v_{j,p} + u_{j,p}}{q_p}\right) (EA_{j,p}) = 1 + \frac{m}{\sqrt{D'}} K_S^2.$$

Since $0 < \frac{m}{\sqrt{D'}} K_S^2 \leq \frac{1}{12}$, we have

$$1 < \sum_{p \neq p_4} \sum_{j=1}^{l_p} \left(1 - \frac{v_{j,p} + u_{j,p}}{q_p}\right) (EA_{j,p}) \leq 1 + \frac{1}{12}.$$

It is easy to see that Table 8 contains no solution to this inequality. \square

Now we have 4 cases

- (1) $E \cdot f^{-1}(p_i) = 0$ for $i = 1, 2, 3$, and E meets one component of $f^{-1}(p_4)$ with multiplicity 2.
- (2) $E \cdot f^{-1}(p_i) = 0$ for $i = 1, 2, 3$, and E meets two non-end components of $f^{-1}(p_4)$.
- (3) $E \cdot f^{-1}(p_i) = 0$ for $i = 1, 2, 3$, and E meets both end components of $f^{-1}(p_4)$.
- (4) $E \cdot f^{-1}(p_i) = 0$ for $i = 1, 2, 3$, and E meets an end component and a non-end component of $f^{-1}(p_4)$.

Step 3. Case (1) cannot occur.

Proof. Suppose that Case (1) occurs, i.e., $EA_s = 2$ for some $1 \leq s \leq l$, $EA_j = 0$ for $j \neq s$.

If $1 < s < l$, then Proposition 4.2(1),(3) give

$$1 - \frac{m}{\sqrt{D'}} K_S^2 = 2 \left(\frac{v_s + u_s}{q} \right)$$

and

$$1 + \frac{m^2}{D'} K_S^2 = 4 \frac{v_s u_s}{q}.$$

Subtracting the first equality multiplied by 2 from the second, we get

$$\frac{m^2}{D'} K_S^2 + 2 \frac{m}{\sqrt{D'}} K_S^2 - 1 = 4 \frac{v_s u_s}{q} - 4 \left(\frac{v_s + u_s}{q} \right) > 0.$$

On the other hand, by Step 1,

$$\frac{m^2}{D'} K_S^2 + 2 \frac{m}{\sqrt{D'}} K_S^2 - 1 \leq \frac{1}{36} + \frac{2}{12} - 1 < 0,$$

a contradiction.

If $s = 1$, then Proposition 4.2(1),(3) give

$$1 - \frac{m}{\sqrt{D'}} K_S^2 = 2 \left(\frac{v_1 + 1}{q} \right)$$

and

$$1 + \frac{m^2}{D'} K_S^2 = 4 \frac{v_1}{q}.$$

Eliminating $\frac{v_1}{q}$, we get

$$1 = \frac{m^2}{D'} K_S^2 + 2 \frac{m}{\sqrt{D'}} K_S^2 + \frac{4}{q} \leq \frac{1}{36} + \frac{2}{12} + \frac{4}{20} < 1,$$

a contradiction. □

Step 4. Case (2) cannot occur.

Proof. Suppose that Case (2) occurs, i.e., $EA_s = EA_t = 1$ for some $1 < s < t < l$, $EA_j = 0$ for $j \neq s, t$. Proposition 4.2(1),(2) give

$$1 - \frac{m}{\sqrt{D'}} K_S^2 = \frac{v_s + u_s}{q} + \frac{v_t + u_t}{q}$$

and

$$1 + \frac{m^2}{D'} K_S^2 \geq \frac{v_s u_s}{q} + \frac{v_t u_t}{q}.$$

Subtracting the equality multiplied by $\frac{4}{3}$ from the inequality, we get

$$1 + \frac{m^2}{D'} K_S^2 - \frac{4}{3} + \frac{4m}{3\sqrt{D'}} K_S^2 \geq \frac{v_s u_s}{q} + \frac{v_t u_t}{q} - \frac{4}{3} \left(\frac{v_s + u_s}{q} + \frac{v_t + u_t}{q} \right) \geq 0,$$

where the last inequality follows from

$$vu - \frac{4}{3}(v + u) = (v - \frac{4}{3})(u - \frac{4}{3}) - \frac{16}{9} \geq 0$$

for $v \geq 2, u \geq 2, v + u \geq 7$. ($l \geq 6$ implies $v + u \geq 7$.)

Since $\frac{m^2}{D'} K_S^2 + \frac{4m}{3\sqrt{D'}} K_S^2 < \frac{1}{3}$, it gives a contradiction. □

Step 5. Case (3) cannot occur.

Proof. Suppose that Case (3) occurs, i.e., $EA_1 = EA_l = 1$, $EA_j = 0$ for $j \neq 1, l$. Then, by Proposition 4.2 (1), we obtain

$$\frac{q_1 + q_l + 2}{q} = 1 - \frac{m}{\sqrt{D'}} K_S^2.$$

Also by Proposition 4.2 (3), we obtain

$$\frac{q_1 + q_l + 2}{q} = 1 + \frac{m^2}{D'} K_S^2.$$

From these two equations we obtain $m = -\sqrt{D'}$ and hence $-K_S$ is ample by Lemma 3.7(5). \square

Step 6. Case (4) cannot occur.

Proof. Suppose that Case (4) occurs, i.e., $EA_1 = EA_t = 1$ for some $1 < t < l$ and $EA_j = 0$ for $j \neq 1, t$. Proposition 4.2(1),(3) give

$$1 - \frac{m}{\sqrt{D'}} K_S^2 = \frac{q_1 + 1}{q} + \frac{v_t + u_t}{q} = \frac{q_1 - 1}{q} + \frac{v_t + (u_t + 2)}{q}$$

and

$$1 + \frac{m^2}{D'} K_S^2 = \frac{q_1}{q} + \frac{v_t u_t}{q} + 2 \frac{v_t}{q} = \frac{q_1}{q} + \frac{v_t(u_t + 2)}{q}.$$

Subtracting the first equality multiplied by $\frac{3}{2}$ from the second, we get

$$\begin{aligned} 1 + \frac{m^2}{D'} K_S^2 - \frac{3}{2} + \frac{3m}{2\sqrt{D'}} K_S^2 &= \frac{q_1}{q} - \frac{3(q_1 - 1)}{2q} + \frac{v_t(u_t + 2)}{q} - \frac{3}{2} \left(\frac{v_t + (u_t + 2)}{q} \right) \\ &> \frac{q_1}{q} - \frac{3(q_1 - 1)}{2q} = -\frac{q_1 - 3}{2q}, \end{aligned}$$

where the inequality follows from

$$vu' - \frac{3}{2}(v + u') = (v - \frac{3}{2})(u' - \frac{3}{2}) - \frac{9}{4} > 0$$

for $v \geq 2, u' \geq 4, v + u' \geq 9$. ($l \geq 6$ implies $v + u' = v + (u + 2) \geq 9$.) Thus

$$\frac{q_1}{2q} > \frac{q_1 - 3}{2q} > \frac{1}{2} - \frac{m^2}{D'} K_S^2 - \frac{3m}{2\sqrt{D'}} K_S^2 \geq \frac{1}{2} - \frac{1}{36} - \frac{3}{2} \cdot \frac{1}{12} = \frac{25}{72},$$

hence

$$\frac{q_1}{q} > \frac{25}{36} > \frac{1}{2}.$$

It implies, in particular, that

$$n_1 = 2.$$

We claim that $n_t = 2$. Suppose that $n_t > 2$. Let

$$\sigma : S' \rightarrow S''$$

be the blow down of the (-1)-curve E , and

$$g : S'' \rightarrow \bar{S}$$

be the contraction to another \mathbb{Q} -homology projective plane \bar{S} with

$$L_{\bar{S}} := b_2(S'') - 1 = L - 1.$$

Note that \bar{S} has 3 singularities $\bar{p}_1, \bar{p}_2, \bar{p}_3$ of order 2,3,5 of the same type as S , and a singularity \bar{p}_4 of order q' with $q' < q$. The latter follows from Lemma 2.4(6).

Moreover the image \bar{A}_1 on S'' is a (-1) -curve, and the images $\bar{A}_2, \dots, \bar{A}_l$ are the components of $g^{-1}(\bar{p}_4)$.

We claim that $K_{\bar{S}}$ is ample. To prove this, note first that $K_{\bar{S}}$ is ample if and only if the coefficient of \bar{A}_1 in $g^*K_{\bar{S}}$, when written as a linear combination of \bar{A}_1 and g -exceptional curves, is positive. Let C be the coefficient. From the adjunction formula

$$K_{S'} = f^*K_S - \sum \mathcal{D}_{p_i} = \sigma^*(g^*K_{\bar{S}} - \sum \mathcal{D}_{\bar{p}_i}) + E,$$

we see that C is equal to the coefficient of A_1 in $K_{S'}$, when written as a linear combination of E and f -exceptional curves. To compute C , we localize at p_4 and write

$$f^*K_S = xE + \sum (y_j A_j),$$

$$\mathcal{D}_{p_4} = \sum (d_j A_j)$$

for some rational numbers x, y_j, d_j . Then

$$C = y_1 - d_1.$$

Since E is of the form (3.1), it is easy to see

$$x = \frac{\sqrt{D'}}{m}.$$

From the two systems of equations

$$(f^*K_S)A_i = 0, \quad (1 \leq i \leq l),$$

and

$$(\mathcal{D}_{p_4})A_i = -n_i + 2, \quad (1 \leq i \leq l),$$

we get

$$y_1 = \frac{x(q_1 + v_t)}{q}, \quad d_1 = 1 - \frac{q_1 + 1}{q}$$

respectively. Now since $x \geq L - 9 \geq 3$ and $\frac{q_1}{q} > \frac{25}{36}$, we see that

$$C = y_1 - d_1 = \frac{x(q_1 + v_t)}{q} + \frac{q_1 + 1}{q} - 1 \geq \frac{4q_1}{q} + \frac{3v_t + 1}{q} - 1 > 0.$$

This proves that $K_{\bar{S}}$ is ample. If \bar{S} has $L_{\bar{S}} < 12$ or $q' < 20$, then we are done by Lemma 5.5. Otherwise, we can find a (-1) -curve E' on S'' of the form (3.1) with

$$0 < \frac{m}{\sqrt{D'}} \leq \frac{1}{L_{\bar{S}} - 9} \leq \frac{1}{3}.$$

We restart with E' on S'' from Step 1. Then, by Step 1 to Step 5, we may assume that E' satisfies the case (4), i.e., we may assume that $E'\bar{A}_2 = E'\bar{A}_{t'} = 1$ with $2 < t' < l$. Here $\bar{A}_2, \dots, \bar{A}_l$ are the components lying over the singularity \bar{p}_4 . If $-\bar{A}_{t'}^2 > 2$, we repeat the above process. Since each process decreases by 1 the number L , we may assume that $n_t = 2$ at certain stage. Now by Lemma 2.4(3)

$$\frac{u_t v_t}{q} \geq \frac{1}{n_t} = \frac{1}{2}.$$

Thus

$$\frac{37}{36} \geq 1 + \frac{m^2}{D'} K_{\bar{S}}^2 = \frac{q_1}{q} + \frac{u_t v_t + 2v_t}{q} > \frac{q_1}{q} + \frac{u_t v_t}{q} \geq \frac{25}{36} + \frac{1}{2} = \frac{43}{36},$$

a contradiction. \square

This completes the proof of Theorem 1.4.

7. LOG DEL PEZZO SURFACES OF RANK ONE

Throughout this section, S denotes a \mathbb{Q} -homology projective plane with quotient singularities such that $-K_S$ is ample, i.e., S is a log del Pezzo surface of rank one. Let

$$f : S' \rightarrow S$$

be a minimal resolution of S . Let

$$\mathcal{F} := f^{-1}(\text{Sing}(S))$$

be the reduced exceptional divisor of f .

We review the work of Zhang [Z], Gurjar and Zhang [GZ] and Belousov [Be] on log del Pezzo surfaces of rank one.

Lemma 7.1. $B^2 \geq -1$ for any irreducible curve $B \subset S'$ not contracted by $f : S' \rightarrow S$.

Proof. This is well-known (cf. [HK2], Lemma 2.1). \square

Theorem 7.2 ([Be]). S has at most 4 singular points.

The following lemma is given in Lemma 4.1 in [Z], and can also be easily derived from the inequality of Proposition 4.2(1) when S has only cyclic singularities.

Lemma 7.3 ([Z]). Let E be a (-1) -curve on S' . Let A_1, \dots, A_r exhaust all irreducible components of \mathcal{F} such that $EA_i > 0$. Suppose that $A_1^2 \geq A_2^2 \geq \dots \geq A_r^2$. Then the r -tuple $(-A_1^2, \dots, -A_r^2)$ is one of the following:

$$(2, \dots, 2, n), n \geq 2, (2, \dots, 2, 3, 3), (2, \dots, 2, 3, 4), (2, \dots, 2, 3, 5).$$

An irreducible curve C on S' is called a *minimal curve* if $C \cdot (-f^*K_S)$ attains the minimal positive value.

Lemma 7.4 ([Z]). A minimal curve C is a smooth rational curve.

Lemma 7.5 ([Z], Lemma 2.1, [GZ], Remark 3.4). Let C be a minimal curve. Suppose that $|C + \mathcal{F} + K_{S'}| \neq \emptyset$. Then there is a unique decomposition $\mathcal{F} = \mathcal{F}' + \mathcal{F}''$ such that

- (1) \mathcal{F}' consists of (-2) -curves not meeting $C + \mathcal{F}''$,
- (2) $C + \mathcal{F}'' + K_{S'} \sim 0$,
- (3) $\mathcal{F}'' = f^{-1}(p)$ for some singular point p unless $\mathcal{F}'' = 0$.

Furthermore, if $\mathcal{F}'' \neq 0$, then $C \cdot \mathcal{F}'' = C \cdot \mathcal{F} = 2$ and one of the following holds:

- (1) \mathcal{F}'' consists of one irreducible component, which C meets in a single point with multiplicity 2 or in two points,
- (2) \mathcal{F}'' consists of two irreducible components, whose intersection point C passes through,
- (3) \mathcal{F}'' consists of at least two irreducible components, and C meets the two end components of \mathcal{F}'' .

Lemma 7.6 ([GZ], Proposition 3.6). Let C be a minimal curve. Suppose that $|C + \mathcal{F} + K_S| = \emptyset$. Then C is a (-1) -curve, or $S' = S \cong \mathbb{P}^2$ and C is a line, or S is a Hirzebruch surface with the negative section contracted and C is a fibre on the Hirzebruch surface.

Lemma 7.7 ([Be], Lemma 4.1). *Suppose that S' contains a minimal curve C with $C^2 = -1$. Suppose that $|C + \mathcal{F} + K_{S'}| = \emptyset$. Then $CF' \leq 1$ for any connected component \mathcal{F}' of \mathcal{F} .*

Lemma 7.8 ([Z], Lemma 4.4). *Suppose that S' contains a minimal curve C with $C^2 = -1$. Suppose that $|C + \mathcal{F} + K_{S'}| = \emptyset$, and that C meets exactly two components F_1, F_2 of \mathcal{F} . Then either $F_1^2 = -2$ or $F_2^2 = -2$.*

The following lemma was proved in ([Z], the proof of Lemma 5.3).

Lemma 7.9. *With the same assumption as in Lemma 7.8, assume further that $F_1^2 = F_2^2 = -2$. If F_1 is not an end component, then one of the following two cases holds:*

- (1) *There exists another minimal (-1) -curve C' such that $|C' + \mathcal{F} + K_{S'}| \neq \emptyset$.*
- (2) *$F_2 = f^{-1}(p_i)$ for some singular point p_i .*

Lemma 7.10. *Suppose that S' contains a minimal curve C with $C^2 = -1$. Suppose that $|C + \mathcal{F} + K_{S'}| = \emptyset$, and that C meets three components F_1, F_2, F_3 of \mathcal{F} and possibly more. Define*

$$G := 2C + F_1 + F_2 + F_3 + K_{S'}.$$

Then either $G \sim 0$ or $G \sim \Gamma$ for some (-1) -curve Γ such that $C\Gamma = F_i\Gamma = 0$ for $i = 1, 2, 3$. Furthermore, the following hold true.

- (1) *In the first case, there are 3 singular points p_1, p_2, p_3 such that $f^{-1}(p_i) = F_i$, and C meets no component of $\mathcal{F} - (F_1 + F_2 + F_3)$.*
- (2) *In the second case,*
 - (a) *$L = 2 - (F_1^2 + F_2^2 + F_3^2)$, where L is the number of irreducible components of \mathcal{F} ,*
 - (b) *each curve in $\mathcal{F} - F_1 - F_2 - F_3$ is a (-2) - or a (-3) -curve and there are at most two (-3) -curves in $\mathcal{F} - F_1 - F_2 - F_3$,*
 - (c) *each connected component of \mathcal{F} contains at most one $(-n)$ -curve with $n \geq 3$.*

Proof. The main assertion is exactly ([Z], Lemma 2.3).

(1) Let F_i be an irreducible component of $f^{-1}(p_i)$. Suppose that $f^{-1}(p_i)$ has at least 2 irreducible components. Then there is an irreducible component I of $f^{-1}(p_i)$ such that $IF_i = 1$. By Lemma 7.7, $IC = 0$, hence

$$0 = IG = I.(2C + F_1 + F_2 + F_3 + K_{S'}) = IF_i + IK_{S'} = 1 - I^2 - 2.$$

Thus $I^2 = -1$, a contradiction.

Suppose that C meets a component J of $\mathcal{F} - (F_1 + F_2 + F_3)$. Then

$$0 = JG = J.(2C + F_1 + F_2 + F_3 + K_{S'}) = 2 + JK_{S'},$$

so $J^2 = 0$, a contradiction.

(2-a) We note that

$$G^2 = (2C + F_1 + F_2 + F_3 + K_{S'})^2 = 1 - L - (F_1^2 + F_2^2 + F_3^2).$$

Since $G^2 = \Gamma^2 = -1$, we have $L = 2 - (F_1^2 + F_2^2 + F_3^2)$.

(2-b) and (2-c) are exactly ([GZ], Lemma 6.6). □

The following lemma was proved in ([Z], the proof of Lemma 5.2).

Lemma 7.11. *With the same assumption as in Lemma 7.10, assume further that $2C + F_1 + F_2 + F_3 + K_{S'} \sim \Gamma$ for some (-1) -curve Γ , and that at least two of F_1, F_2, F_3 are (-2) -curves. Then one of the following two cases holds:*

- (1) *There exists another minimal (-1) -curve C' such that $|C' + \mathcal{F} + K_{S'}| \neq \emptyset$.*
- (2) *S has a non-cyclic singularity.*

8. PROOF OF THEOREM 1.6

Let S be a \mathbb{Q} -homology projective plane with cyclic quotient singularities such that

- $H_1(S^0, \mathbb{Z}) = 0$,
- $-K_S$ is ample.

Assume that S has exactly 4 cyclic singularities p_1, p_2, p_3, p_4 . In Section 5, we have enumerated all possible 4-tuples of orders of local fundamental groups:

- (1) $(2, 3, 5, q)$, $q \geq 7$, $\gcd(q, 30) = 1$,
- (2) $(2, 3, 7, q)$, $11 \leq q \leq 41$, $\gcd(q, 42) = 1$,
- (3) $(2, 3, 11, 13)$.

For (2) and (3), we have seen that there are 24 different possible types for R , the sublattice of $H^2(S', \mathbb{Z})$ generated by all exceptional curves of the minimal resolution $f : S' \rightarrow S$, as shown in Table 1.

For (1), the order 3 singularity is of type $\frac{1}{3}(1, 1)$ (Lemma 5.3), so it remains to consider the following cases:

- $A_1 + \frac{1}{3}(1, 1) + \frac{1}{5}(1, 1) + \frac{1}{q}(1, q_1)$, $q \geq 7$, $\gcd(q, 30) = 1$;
- $A_1 + \frac{1}{3}(1, 1) + \frac{1}{5}(1, 2) + \frac{1}{q}(1, q_1)$, $q \geq 7$, $\gcd(q, 30) = 1$;
- $A_1 + \frac{1}{3}(1, 1) + A_4 + \frac{1}{q}(1, q_1)$, $q \geq 7$, $\gcd(q, 30) = 1$;
- the 24 cases in Table 1.

Let

$$\mathcal{F} = f^{-1}(\text{Sing}(S))$$

be the reduced exceptional divisor of the minimal resolution $f : S' \rightarrow S$.

Let C be a (fixed) minimal curve on S' . Since $-K_S$ is ample, by Lemma 3.7, C can be written as

$$(8.1) \quad C = -mM + \sum_{p \in \text{Sing}(S)} \sum_{i=1}^{l_p} a_{i,p} A_{i,p}$$

for some integer $m > 0$ and some $a_{i,p} \in \frac{1}{c}\mathbb{Z}$.

8.1. Step 1. $|C + \mathcal{F} + K_{S'}| = \emptyset$.

Proof. Suppose that $|C + \mathcal{F} + K_{S'}| \neq \emptyset$. By Lemma 7.5, we see that S has at least 3 rational double points.

In the case of $(2, 3, 5, q)$, by Lemma 5.3 we see that S has 3 rational double points, only if the singularities are of type $A_1 + [3] + A_4 + A_{q-1}$. In this case,

$$L = q + 5 \text{ and } K_S^2 = 9 - (q + 5) + \frac{1}{3} < 0,$$

a contradiction.

We also see that each of the 24 cases from Table 1 has at most 2 rational double points. \square

8.2. Step 2.

- (1) C is a (-1) -curve.
- (2) $C\mathcal{F} = 3$, and C meets three distinct components F_1, F_2, F_3 of \mathcal{F} .

Proof. (1) It immediately follows from Lemma 7.6 since S has 4 singularities.

(2) By Lemma 7.7, $C\mathcal{F} \leq 4$. Since $C^2 = -1 < 0$ and the lattice R is negative definite, $C\mathcal{F} \geq 1$.

Assume that $C\mathcal{F} = 1$. Blowing up the intersection point, then contracting the proper transform of C and the proper transforms of all irreducible components of \mathcal{F} , we obtain a \mathbb{Q} -homology projective plane with 5 quotient singularities, which contradicts the result of [HK1] since S is rational.

Assume that $C\mathcal{F} = 4$. By Lemma 7.7, C meets four components F_1, F_2, F_3, F_4 of \mathcal{F} , where $F_i \subset f^{-1}(p_i)$. Then $G \sim \Gamma$ by Lemma 7.10 (1). By Lemma 7.3, at least two of F_1, F_2, F_3, F_4 have self-intersection -2 . Thus, by Lemma 7.11, there exists another minimal (-1) -curve C' such that $|C' + \mathcal{F} + K_{S'}| \neq \emptyset$. This is impossible by Step 1.

Assume that $C\mathcal{F} = 2$.

(a) Suppose that the case $(2, 3, 5, q)$ occurs for some $q \geq 7$ with $\gcd(q, 30) = 1$. By Lemma 5.4, $C.f^{-1}(p_4) = 2$. But, By Lemma 7.7, $C.f^{-1}(p_4) \leq 1$, a contradiction.

(b) Now suppose that one of the 24 cases of Table 1 occurs. By Lemma 7.7, there are two components F_1 and F_2 of \mathcal{F} with $CF_1 = CF_2 = 1$. By Lemma 7.8, we may assume that $F_1^2 = -2$. Moreover, by Lemma 5.6, C does not meet an end component of $f^{-1}(p_i)$ for any i , i.e., both F_1 and F_2 are middle components. Thus $F_2^2 \neq -2$ by Lemma 7.9 and Step 1. After contracting the (-1) -curve C , by contracting the proper transforms of all irreducible components of $\mathcal{F} - F_1$, we obtain a \mathbb{Q} -homology projective plane with 5 quotient singularities, which contradicts the result of [HK1] since S is rational. \square

8.3. Step 3. $2C + F_1 + F_2 + F_3 + K_{S'} \sim \Gamma$ for some (-1) -curve Γ .

Proof. Suppose that

$$2C + F_1 + F_2 + F_3 + K_{S'} \sim 0.$$

Then, by Lemma 7.10(1), each F_i is equal to the inverse image of a singular point of S . By Table 1 and Lemma 5.3, only the following cases satisfy this condition:

$$\begin{aligned} & A_1 + A_2 + [7] + [13] && (\text{Case 1, Table 1}), \\ & A_1 + [3] + [2, 2, 2, 2] + [q], \\ & A_1 + [3] + [3, 2] + [q], \\ & A_1 + [3] + [5] + \frac{1}{q}(1, q_1). \end{aligned}$$

Thus,

$$(-F_1^2, -F_2^2, -F_3^2) = (2, 7, 13), (2, 3, q), (2, 5, q), (3, 5, q), (2, 3, 5).$$

Then Lemma 7.3 rules out the first four possibilities.

In the last case $(-F_1^2, -F_2^2, -F_3^2) = (2, 3, 5)$, $F_i = f^{-1}(p_i)$ for $i = 1, 2, 3$. In this case we consider the sublattice

$$\langle C, F_1, F_2, F_3 \rangle \subset H^2(S', \mathbb{Z})$$

generated by C, F_1, F_2, F_3 . It is of rank 4 and has

$$\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -3 & 0 \\ 1 & 0 & 0 & -5 \end{pmatrix}$$

as its intersection matrix. It has determinant -1 , hence the orthogonal complement of $\langle C, F_1, F_2, F_3 \rangle$ in $H^2(S', \mathbb{Z})$ is unimodular. The orthogonal complement is an over-lattice of the lattice R_{p_4} generated by the components of $f^{-1}(p_4)$. Since R_{p_4} is a primitive sublattice of $H^2(S', \mathbb{Z})$, it must be unimodular, hence $q = 1$, a contradiction. \square

8.4. Step 4. If one of the cases $(2, 3, 5, q)$, $q \geq 7$, $\gcd(q, 30) = 1$, occurs, then $C.f^{-1}(p_4) = 1$.

Proof. Suppose that the case $(2, 3, 5, q)$ occurs for some $q \geq 7$ with $\gcd(q, 30) = 1$. By Lemma 5.3, p_2 is of type [3].

By Lemma 7.7, $C.f^{-1}(p_i) \leq 1$ for $i = 1, 2, 3, 4$.

Suppose on the contrary that $C.f^{-1}(p_4) = 0$.

Then,

$$C.f^{-1}(p_1) = C.f^{-1}(p_2) = C.f^{-1}(p_3) = 1.$$

Let $F_i \subset f^{-1}(p_i)$ be the component with $CF_i = 1$ for $i = 1, 2, 3$.

Assume that p_3 is of type [5]. Then $(-F_1^2, -F_2^2, -F_3^2) = (2, 3, 5)$ and the sublattice $\langle C, F_1, F_2, F_3 \rangle \subset H^2(S', \mathbb{Z})$ has determinant -1 , leading to the same contradiction as above, since the orthogonal complement of $\langle C, F_1, F_2, F_3 \rangle$ in $H^2(S', \mathbb{Z})$ is R_{p_4} .

Assume that p_3 is of type [2, 3]. Then $(-F_1^2, -F_2^2, -F_3^2) = (2, 3, 2)$ or $(2, 3, 3)$. Let $f^{-1}(p_3) = F_3 + F'_3$. If $F_3^2 = -2$, then

$$|\det \langle C, F_1, F_2, F_3, F'_3 \rangle| = 13,$$

and by Lemma 7.10(2-a) $L = 2 + 2 + 3 + 2 = 9$, so $l = 5$. The orthogonal complement of $\langle C, F_1, F_2, F_3, F'_3 \rangle$ in $H^2(S', \mathbb{Z})$ is R_{p_4} , hence

$$|\det(R_{p_4})| = q = 13.$$

This leads to a contradiction since there is no continued fraction of length 5 with $q = 13$. If $F_3^2 = -3$, then

$$|\det \langle C, F_1, F_2, F_3, F'_3 \rangle| = 7,$$

hence $|\det(R_{p_4})| = q = 7$. By Lemma 7.10(2), $L = 2 + 2 + 3 + 3 = 10$, so $l = 6$. Thus p_4 is of type A_6 . But, then

$$K_S^2 = 9 - L - \mathcal{D}_{p_2}^2 - \mathcal{D}_{p_3}^2 = -1 + \frac{1}{3} + \frac{2}{5} < 0,$$

a contradiction.

Assume that p_3 is of type $A_4 = [2, 2, 2, 2]$. Then $(-F_1^2, -F_2^2, -F_3^2) = (2, 3, 2)$. Let $f^{-1}(p_3) = H_1 + H_2 + H_3 + H_4$. If F_3 is an end component of $f^{-1}(p_3)$, say H_1 , then

$$|\det \langle C, F_1, F_2, H_1, H_2, H_3, H_4 \rangle| = 19,$$

and by Lemma 7.10(2-a) $L = 2 + 2 + 3 + 2 = 9$, so $l = 3$. Thus $|\det(R_{p_4})| = q = 19$ and $\text{rank}(R_{p_4}) = 3$. Among all Hirzebruch-Jung continued fractions of order 19, only two, $[7, 2, 2]$ and $[3, 4, 2]$, have length 3. In each of these two cases, $f^{-1}(p_4)$

contains an irreducible component with self-intersection ≤ -4 . Since $f^{-1}(p_4) \subset \mathcal{F} - F_1 - F_2 - F_3$, we have a contradiction by Lemma 7.10(2-b). If F_3 is a middle component of $f^{-1}(p_3)$, say H_2 , then

$$|\det(C, F_1, F_2, H_1, H_2, H_3, H_4)| = 31,$$

and by Lemma 7.10(2-a) $L = 2 + 2 + 3 + 2 = 9$, so $l = 3$. Thus $q = 31$ and p_4 is of type $[11, 2, 2]$, $[3, 6, 2]$, or $[5, 2, 4]$. In each of these three cases, $f^{-1}(p_4)$ contains an irreducible component with self-intersection ≤ -4 , a contradiction by Lemma 7.10(2-b). This proves that $C \cdot f^{-1}(p_4) = 1$. \square

8.5. Step 5. None of the cases $(2, 3, 5, q)$, $q \geq 7$, $\gcd(q, 30) = 1$, occurs.

Proof. Suppose that the case $(2, 3, 5, q)$ occurs for some $q \geq 7$ with $\gcd(q, 30) = 1$.

By Lemma 5.3, p_2 is of type $[3]$.

By Step 2, $C\mathcal{F} = 3$ and C meets the three components F_1, F_2, F_3 of \mathcal{F} .

By Step 3,

$$2C + F_1 + F_2 + F_3 + K_{S'} \sim \Gamma$$

for some (-1) -curve Γ .

By Step 4, we may assume that $F_3 \subset f^{-1}(p_4)$.

Let

$$f^{-1}(p_4) = \frac{-n_1}{D_1} - \frac{-n_2}{D_2} - \dots - \frac{-n_l}{D_l}$$

and $F_3 = D_j$ for some $1 \leq j \leq l$. Note first that by Lemma 7.10(2-b), $n_k \leq 3$ for all $k \neq j$.

Assume that p_3 is of type $[5]$. By Lemma 7.10(2-b), C must meet $f^{-1}(p_3)$, so we may assume that $F_2 = f^{-1}(p_3)$. Since $F_1 = f^{-1}(p_1)$ or $F_1 = f^{-1}(p_2)$, by Lemma 7.3,

$$(-F_1^2, -F_2^2, -F_3^2) = (2, 5, 2), (3, 5, 2), (2, 5, 3).$$

By Lemma 7.10 (2-a), we have

$$(L, n_j) = (11, 2), (12, 2), (12, 3),$$

hence

$$(l, n_j) = (8, 2), (9, 2), (9, 3).$$

By Lemma 7.10 (2-b) and (2-c),

$$[n_1, \dots, n_l] = \begin{array}{l} [3, 2, 2, 2, 2, 2, 2, 2], [2, 2, 2, 2, 2, 2, 2, 2]; \\ [3, 2, 2, 2, 2, 2, 2, 2], [2, 2, 2, 2, 2, 2, 2, 2] \end{array}$$

up to permutation of n_1, \dots, n_l . Counting all possible permutations and identifying $[n_1, \dots, n_l]$ with its reverse $[n_l, \dots, n_1]$, it is easy to see that there are

$$4 + 1 + 5 + 1 = 11$$

possible cases for $[n_1, \dots, n_l]$.

E.g., $[3, 2, 2, 2, 2, 2, 2, 2]$ gives 4 possible cases for $[n_1, \dots, n_l]$. None of these 11 cases satisfies the following three conditions:

- (#1) $K_S^2 > 0$,
- (#2) $\gcd(q, 30) = 1$,
- (#3) $D = |\det(R)|K_S^2$ is a positive square integer.

Assume that p_3 is of type $[2, 3]$. Then, by Lemma 7.3,

$$(-F_1^2, -F_2^2, -F_3^2) = (2, 3, n_j), n_j \leq 5, \text{ or } (3, 3, n_j), n_j = 2, \text{ or } (2, 2, n_j).$$

The last case can be ruled out by Lemma 7.11 and Step 1 since S has only cyclic singularities. Now, by Lemma 7.10(2), we have

$$(l, n_j) = (5, 2), (6, 3), (7, 4), (8, 5), (6, 2),$$

and

$$\begin{aligned} [n_1, \dots, n_l] = & [3, 2, 2, 2, 2], [2, 2, 2, 2, 2]; [3, 2, 2, 2, 2, 2]; \\ & [4, 2, 2, 2, 2, 2, 2]; [5, 2, 2, 2, 2, 2, 2, 2]; [2, 2, 2, 2, 2, 2, 2], \end{aligned}$$

up to permutation of n_1, \dots, n_l . It is easy to see that there are 16 possible cases for $[n_1, \dots, n_l]$. None of them satisfies the three conditions $(\#1), (\#2), (\#3)$.

Assume that p_3 is of type $[2, 2, 2, 2]$. Then, by Lemma 7.3,

$$(-F_1^2, -F_2^2, -F_3^2) = (2, 3, n_j), n_j \leq 5, \text{ or } (2, 2, n_j).$$

The last case can be ruled out by Lemma 7.11 and Step 1 since S has only cyclic singularities. Now, by Lemma 7.10(2), we have

$$(l, n_j) = (3, 2), (4, 3), (5, 4), (6, 5),$$

and

$$[n_1, \dots, n_l] = [3, 2, 2], [2, 2, 2]; [3, 2, 2, 2]; [4, 2, 2, 2, 2]; [5, 2, 2, 2, 2, 2],$$

up to permutation of n_1, \dots, n_l . It is easy to see that there are 11 possible cases for $[n_1, \dots, n_l]$. None of them satisfies the three conditions $(\#1), (\#2), (\#3)$. \square

Next, we will show that none of the cases $(2, 3, 7, q)$, $11 \leq q \leq 41$, $\gcd(q, 42) = 1$, and $(2, 3, 11, 13)$ occurs. To do this, it is enough to consider the 24 cases of Table 1.

8.6. Step 6. None of the 24 cases of Table 1 occurs.

Proof. By Step 2, $C\mathcal{F} = 3$ in each of the 24 cases of Table 1.

Each of Cases (1), (2), (3), (4), (6), (8), (9), (11), (12), (13), (17), and (19), contains an irreducible components F' with self-intersection ≤ -6 . Lemma 7.10 (2-b) implies that C meets F' . Thus C meets two components of \mathcal{F} with self-intersection -2 by Lemma 7.3. Thus we get a contradiction for those cases by Lemma 7.11 and Step 1.

By Lemma 7.10 (2-c), we get a contradiction immediately for Cases (7), (10), (14), (16), (18), since each of these cases contains a connected component of \mathcal{F} with at least two irreducible components of self-intersection ≤ -3 .

By Lemma 7.3 and Lemma 7.10 (2-b), we get a contradiction immediately for Cases (5), (20), (21), (22), since each of these cases contains at least two irreducible components with self-intersection ≤ -4 .

We need to rule out the remaining three cases: (15), (23), (24).

Consider Case (24). Note that $L = 10$ in this case. On the other hand, by Lemma 7.10(2-b), C must meet the component having self-intersection number -5 . Thus, we may assume that $F_3^2 = -5$. Since $F_1^2 \leq -2, F_2^2 \leq -2$, Lemma 7.10 (2-a) gives $L = 2 - (F_1^2 + F_2^2 + F_3^2) \geq 2 + 2 + 2 + 5 = 11$, a contradiction.

Case (15): Let

$$\begin{array}{ccccc} \overset{-2}{\underset{\circ}{A}} & \overset{-3}{\underset{\circ}{B}} & \overset{-3}{\underset{\circ}{C_1}} - \overset{-2}{\underset{\circ}{C_2}} - \overset{-2}{\underset{\circ}{C_3}} & \overset{-3}{\underset{\circ}{D_1}} - \overset{-2}{\underset{\circ}{D_2}} - \overset{-2}{\underset{\circ}{D_3}} - \overset{-2}{\underset{\circ}{D_4}} - \overset{-2}{\underset{\circ}{D_5}} \end{array}$$

be the exceptional curves. In this case, $K_S^2 = \frac{50}{231}$, $\sqrt{D} = 10$.

Since $L = 10 = 2 - (F_1^2 + F_2^2 + F_3^2)$, C meets only two of B, C_1, D_1 .

If $CC_1 = CD_1 = 1$, then $CA = 1$. Applying Proposition 4.2(1) to C of the form (8.1) and looking at Table 9, we get

$$\frac{m}{\sqrt{D}} K_S^2 = 1 - \frac{3}{7} - \frac{5}{11} = \frac{9}{77},$$

thus $m = \frac{27}{5}$, not an integer, a contradiction.

TABLE 9.

	[2]	[3]	[3, 2, 2]			[3, 2, 2, 2, 2]				
j	1	1	1	2	3	1	2	3	4	5
$1 - \frac{v_j + u_j}{q}$	0	$\frac{1}{3}$	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{1}{7}$	$\frac{5}{11}$	$\frac{4}{11}$	$\frac{3}{11}$	$\frac{2}{11}$	$\frac{1}{11}$

If $CB = CC_1 = CA = 1$, then Γ meets C_2 and D_1 only, a contradiction to Lemma 5.6.

If $CB = CC_1 = CD_j = 1$ for some j , then Proposition 4.2(1) gives

$$\frac{m}{\sqrt{D}} K_S^2 = 1 - \frac{1}{3} - \frac{3}{7} - \left(1 - \frac{v_j + u_j}{q}\right) > 0,$$

hence $j = 4, 5$. If $j = 4$, then

$$\frac{m}{\sqrt{D}} K_S^2 = 1 - \frac{1}{3} - \frac{3}{7} - \frac{2}{11} = \frac{13}{231},$$

thus $m = \frac{13}{5}$, a contradiction. If $j = 5$, then

$$\frac{m}{\sqrt{D}} K_S^2 = 1 - \frac{1}{3} - \frac{3}{7} - \frac{1}{11} = \frac{34}{231},$$

thus $m = \frac{34}{5}$, a contradiction.

If $CB = CD_1 = CA = 1$, then

$$\frac{m}{\sqrt{D}} K_S^2 = 1 - \frac{1}{3} - \frac{5}{11} = \frac{7}{33},$$

thus $m = \frac{49}{5}$, a contradiction.

If $CB = CD_1 = CC_2 = 1$, then

$$\frac{m}{\sqrt{D}} K_S^2 = 1 - \frac{1}{3} - \frac{2}{7} - \frac{5}{11} = -\frac{17}{231} < 0,$$

a contradiction.

If $CB = CD_1 = CC_3 = 1$, then

$$\frac{m}{\sqrt{D}} K_S^2 = 1 - \frac{1}{3} - \frac{1}{7} - \frac{5}{11} = \frac{16}{231},$$

thus $m = \frac{16}{5}$, a contradiction.

Case (23): Let

$$\begin{array}{cccccccc} \frac{-2}{\circ} & \frac{-3}{\circ} & \frac{-3}{\circ} & \frac{-2}{\circ} & \frac{-2}{\circ} & \frac{-2}{\circ} & \frac{-2}{\circ} & \frac{-4}{\circ} \\ A & B & C_1 & C_2 & C_3 & C_4 & C_5 & D_1 \\ \frac{-2}{\circ} & & & & & & & \frac{-2}{\circ} \\ & & & & & & & D_2 \\ & & & & & & & \frac{-2}{\circ} \\ & & & & & & & D_3 \\ & & & & & & & \frac{-2}{\circ} \\ & & & & & & & D_4 \end{array}$$

be the exceptional curves. Since C meets D_1 and $L = 11$, C must meet only one of B and C_1 .

If $CB = CA = 1$, then Γ meets exactly two irreducible components C_1, D_2 with multiplicity 1, a contradiction to Lemma 5.6.

If $CB = CC_j = 1$ for some $j \geq 2$, then Table 10 gives

$$\frac{m}{\sqrt{D}} K_S^2 \leq 1 - \frac{1}{3} - \frac{1}{11} - \frac{8}{13} < 0,$$

a contradiction.

If $CC_1 = 1$, then $CA = 1$ and Proposition 4.2(1) together with Table 10 gives

$$\frac{m}{\sqrt{D}} K_S^2 = 1 - 0 - \frac{5}{11} - \frac{8}{13} < 0,$$

a contradiction.

TABLE 10.

	[2]	[3]	[3, 2, 2, 2, 2]					[4, 2, 2, 2]			
j	1	1	1	2	3	4	5	1	2	3	4
$1 - \frac{v_j + u_j}{q}$	0	$\frac{1}{3}$	$\frac{5}{11}$	$\frac{4}{11}$	$\frac{3}{11}$	$\frac{2}{11}$	$\frac{1}{11}$	$\frac{8}{13}$	$\frac{6}{13}$	$\frac{4}{13}$	$\frac{2}{13}$

□

This completes the proof of Theorem 1.6.

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SCHOOL OF MATHEMATICS, KOREA INSTITUTE FOR ADVANCED STUDY, SEOUL 130-722, KOREA
E-mail address: dshwang@kias.re.kr

SCHOOL OF MATHEMATICS, KOREA INSTITUTE FOR ADVANCED STUDY, SEOUL 130-722, KOREA
E-mail address: jhkeum@kias.re.kr